MANAGEMENT SCIENCE (MBA 206)

Lesson Plan

1. Introduction to Management Science
2. Linear Programming- Graphical Method
3. Linear programming- Simplex Method
4. Dual Linear Programming Problems Transportation Problem
5. Sensitivity Analysis in Linear Programming Problems
6. Transportation Model
7. Assignment Model
8. Queuing Theory
9. Inventory Management
10. Network Design, Critical Path Method and PERT
11. Calculation of Float in Network Diagramme
12. Game Theory
13. Decision Theory
14. Integer Programming
15. Goal programming

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BUSINESS ANALYTICS

Course Code: MGT4210 **Credit Units: 02**

Course Objective:

The course provides an introduction to data analytics to be used in business. The students will learn how data analysts describe, predict and make informed business decisions in various business domains like marketing, human resources, finance and operations.The aim of the course is to develop basic data literacy and an analytic mindset in students that will help them to make strategic decisions based on data.

Course Contents:

Module I: Introduction to Business Analytics

Importance and role of data driven decisions. Business Analytics – Definition, Market, Trends; Paradigm Shift from Data to Insight and from Business Intelligence to Business Analytics; Examples and Types of Business Analytics Analysis- Forecasting & Predictive Modeling; Descriptive, Prescriptive and Predictive Analytics. Data Summarization, Data visualization – Various visualization techniques, standardized reporting and Pivot Tables – Using Excel

Module II: Data Mining

Introduction to Data Mining; Crucial processes in data mining; Data Warehousing; Data Mining Techniques and Exploratory Data Analysis; Data Mining Tool – XL Miner.

Module III: Decision Making & Optimization

Decision making under uncertainty – Decision Trees and Risk Profiles; Sensitivity Analysis;
Optimizing complex decisions – Optimization of a large number of decisions while accounting for different kinds of physical and business decisions. Introduction to Optimization Techniques –Linear Programming; Optimization – Use of Excel to solve business problems like marketing mix, capital budgeting and portfolio optimization.

Module IV: Big Data and Introduction to R

Introduction to Big Data, Big Data driven decisions in business organizations – Benefits and Security/Privacy concerns.
Building Business and Economic Models –Tools to leverage data for Prediction purposes; Logistic Regression.
Introduction to Machine Learning; Statistical Learning vs. Machine Learning; Major classes of Learning Algorithms –Supervised Vs Unsupervised Learning.
Introduction to R Programing

Module V: Simulation using R and Excel

Hands on Regression using R; Introduction to Simulation; Applications of Simulation and Building a Simulation Model. (Using Excel and R)
Capstone Project.

Examination Scheme:

Components	**CPA**	**TP**	**Q/S**	**A**	**ME**	**EE**
Weightage (%)	5	5	5	5	10	70

Text & References:

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- Cooper, R.D., Schindler, S. P. and Sharma, J.K. (2015). Business Research Methods. New Delhi, India: McGraw Hill Education (India) Private Limited
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- Kothari, C.R. (2009). *Research Methodology: Methods and Techniques (2nd revised ed.)*. New Delhi, India: New Age International Publisher
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- Data, data everywhere, "Special report on managing information,Economist", February 27th, 2010.
- Liberatore and Luo, "The Analytics Movement, Interfaces, Articles inAdvance", pp. 1–12, 2010.
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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.: 1	Vetter:
Introduction to Management Science	

Structure

1.1 Introduction

1.2 History of Management Science/ Operations Research (OR)

1.3 Stages of Development of Operations Research/ Management Science

1.4 Relationship Between Manager and OR Specialist

1.5 OR Tools and Techniques

1.6 Applications of Operations Research/ Management Science

1.7 Limitations of Operations Research/ Management Science

1.8 Check your progress

1.9 Summary

1.10 Keywords

1.11 Self Assessment Test

1.12 Answers to check your progress

1.13 References/ Suggested Readings

Learning Objectives

After Studying this lesson, students will be able to:

- ❖ Understand the meaning, purpose, and tools of Operations Research
- ❖ Describe the history of Operations Research
- ❖ Describe the Stages of O.R
- ❖ Explain the Applications of Operations Research
- ❖ Describe the Limitations of Operation Research
- ❖ Understand the OR specialist and Manager relationship

1.1 Introduction

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" - which is the term we will use. Another term which is used for this field is "management science" (MS). The Americans sometimes combine the terms OR and MS together and say "OR/MS" or "ORMS". Yet other terms sometimes used are "industrial engineering" (IE) and "decision science" (DS). In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

Operation Research is a relatively new discipline. The contents and the boundaries of the OR are not yet fixed. Therefore, to give a formal definition of the term Operations Research is a difficult task. The OR starts when mathematical and quantitative techniques are used to substantiate the decision being taken. The main activity of a manager is the decision making. In our daily life we make the decisions even without noticing them. The decisions are taken simply by common sense, judgment and expertise without using any mathematical or any other model in simple situations. But the decision we are concerned here with are complex and heavily responsible. Examples are public transportation network planning in a city having its own layout of factories, residential blocks or finding the appropriate product mix when there are a large number of products with different profit contributions and production requirement etc.

Operations Research tools are not from any one discipline. Operations Research takes tools from different discipline such as mathematics, statistics, economics, psychology, engineering etc. and combines these tools to make a new set of knowledge for decision making. Today, O.R. became a professional discipline which deals with the application of scientific methods for making decision, and especially to the allocation of scarce resources. The main purpose of O.R. is to provide a rational basis for decisions making in the absence of complete information, because the systems composed of human, machine, and procedures may do not have complete information.

Operations Research can also be treated as science in the sense it describing, understanding and predicting the systems behaviour, especially man-machine system. Thus O.R. specialists are involved in three classical aspect of science, they are as follows:

i) Determining the systems behaviour
ii) Analyzing the systems behaviour by developing appropriate models

iii) Predict the future behaviour using these models

The emphasis on analysis of operations as a whole distinguishes the O.R. from other research and engineering. O.R. is an interdisciplinary discipline which provided solutions to problems of military operations during World War II, and also successful in other operations. Today business applications are primarily concerned with O.R. analysis for the possible alternative actions. The business and industry befitted from O.R. in the areas of inventory, reorder policies, optimum location and size of warehouses, advertising policies, etc.

As stated earlier defining O.R. is a difficult task. The definitions stressed by various experts and Societies on the subject together enable us to know what O.R. is, and what it does. They are as follows:

1. According to the Operational Research Society of Great Britain (OPERATIONAL RESEARCH QUARTERLY, l3(3):282, l962), Operational Research is the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defense. Its distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as change and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically;
2. Randy Robinson stresses that Operations Research is the application of scientific methods to improve the effectiveness of operations, decisions and management. By means such as analyzing data, creating mathematical models and proposing innovative approaches, Operations Research professionals develop scientifically based information that gives insight and guides decision- making. They also develop related software, systems, services and products;
3. Morse and Kimball have stressed O.R. is a quantitative approach and described it as a "scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control"
4. Saaty considers O.R. as tool of improving quality of answers. He says, "O.R. is the art of giving bad answers to problems which otherwise have worse answers";
5. Miller and Starr state, "O.R. is applied decision theory, which uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough-going rationality in dealing with his decision problem";

6. Pocock stresses that O.R. is an applied Science. He states “O.R. is scientific methodology (analytical, mathematical, and quantitative) which by assessing the overall implication of various alternative courses of action in a management system provides an improved basis for management decisions”.

1.2 History of Operations Research/ Management Science

Operation Research is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study Operation Research, indeed the term O.R. did not exist then. It was really only in the late 1930's that operational research began in a systematic fashion, and it started in the UK. As such it would be interesting to give a short history of O.R.

1936

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

1937

The first of three major pre-war air-defence exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

1938

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the outbreak of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical - aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

1939

In the summer of 1939 Britain held what was to be its last pre-war air defence exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR team was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore in north London.

Initially, they were designated the "Stanmore Research Section". In 1941 they were re-designated the "Operational Research Section" when the term was formalized and officially accepted, and similar sections set up at other RAF commands.

1940

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyses a French request for ten additional fighter squadrons (12 aircraft a squadron - so 120 aircraft in all) when losses were running at some three squadrons every two days (i.e. 36 aircraft every 2 days). They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

1941 onward

In 1941, an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

The responsibility of Coastal Command was, to a large extent, the flying of long-range sorties by single aircraft with the object of sighting and attacking surfaced U-boats (German submarines). The technology of the time meant that (unlike modern day submarines) surfacing was necessary to recharge batteries, vent the boat of fumes and recharge air tanks. Moreover U-boats were much faster on the surface than underwater as well as being less easily detected by sonar.

Thus the Operation Research started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques. Following the end of the war OR spread, although it spread in different ways in the UK and USA.

In 1951 a committee on Operations Research formed by the National Research Council of USA, and the first book on “Methods of Operations Research”, by Morse and Kimball, was published. In 1952 the Operations Research Society of America came into being.

Success of Operations Research in army attracted the attention of the industrial mangers who were seeking solutions to their complex business problems. Now a days, almost every organization in all countries has staff applying operations research, and the use of operations research in government has spread from military to wide variety of departments at all levels. The growth of operations research has not limited to the U.S.A. and U.K., it has reached many countries of the world.

India was one the few first countries who started using operations research. In India, Regional Research Laboratory located at Hyderabad was the first Operations Research unit established during 1949. At the same time another unit was set up in Defense Science Laboratory to solve the Stores, Purchase and Planning Problems. In 1953, Operations Research unit was established in Indian Statistical Institute, Calcutta, with the objective of using Operations Research methods in National Planning and Survey. In 1955, Operations Research Society of India was formed, which is one of the first members of International Federation of Operations Research societies. Today Operations Research is a popular subject in management institutes and schools of mathematics.

1.3 Stages of Development of Operations Research/ Management Science

The stages of development of O.R. are also known as phases and process of O.R, which has six important steps. These six steps are arranged in the following order:

Step I: Observe the problem environment
Step II: Analyze and define the problem
Step III: Develop a model
Step IV: Select appropriate data input
Step V: Provide a solution and test its reasonableness
Step VI: Implement the solution

Step I: Observe the problem environment

The first step in the process of O.R. development is the problem environment observation. This step includes different activities; they are conferences, site visit, research, observations etc. These activities provide sufficient information to the O.R. specialists to formulate the problem.

Step II: Analyze and define the problem

This step is analyzing and defining the problem. In this step in addition to the problem definition the objectives, uses and limitations of O.R. study of the problem also defined. The outputs of this step are clear grasp of need for a solution and its nature understanding.

Step III: Develop a model

This step develops a model; a model is a representation of some abstract or real situation. The models are basically mathematical models, which describes systems, processes in the form of equations, formula/relationships. The different activities in this step are variables definition, formulating equations etc. The model is tested in the field under different environmental constraints and modified in order to work. Some times the model is modified to satisfy the management with the results.

Step IV: Select appropriate data input

A model works appropriately when there is appropriate data input. Hence, selecting appropriate input data is important step in the O.R. development stage or process. The activities in this step include internal/external data analysis, fact analysis, and collection of opinions and use of computer data banks. The objective of this step is to provide sufficient data input to operate and test the model developed in Step_III.

Step V: Provide a solution and test its reasonableness

This step is to get a solution with the help of model and input data. This solution is not implemented immediately, instead the solution is used to test the model and to find there is any limitations. Suppose if the solution is not reasonable or the behaviour of the model is not proper, the model is updated and modified at this stage. The output of this stage is the solution(s) that supports the current organizational objectives.

At this step the solution obtained from the previous step is implemented. The implementation of the solution involves so many behavioral issues. Therefore, before implementation the implementation authority has to resolve the issues. A properly implemented solution results in quality of work and gains the support from the management.

Step VI: Implement this solution

Table 1-1: The process, process activities, and process output

Process Activities	Process	Process Output
Site visits, Conferences, Observations, Research	Step 1: Observe the problem environment	Sufficient information and support to proceed
Define: Use, Objectives, Limitations	Step 2: Analyze and define the problem	Clear grasp of need for and nature of solution requested
Define interrelationships, Formulate equations, Use known O.R. Model , Search alternate Model	Step 3: Develop a Model	Models that works under stated environmental constraints
Analyze: internal-external data, Use computer data banks	Step 4: Select appropriate data input	Sufficient inputs to operate and test model
Test the model find limitations update the model	Step 5: Provide a solution and test its reasonableness	Solution(s) that support current organizational goals
Resolve behavioral issues Sell the idea Give explanations Management involvement	Step-6 Implement the solution	Improved working and Management support for longer run operation of model

1.4 Relationship between the Manager and O.R. Specialist

The key responsibility of manager is decision making. The role of the O.R. specialist is to help the manager make better decisions. Figure 1-1 explains the relationship between the O.R. specialist and the manager/decision maker.

STEPS IN PROBLEM RECOGNITION, O.R. SPECIALIST or FORMULATION AND SOLUTION	INVOLVEMENT: MANAGER
Recognize from organizational symptoms that a problem	Manager
↓ Decide what variables are involved; state the problem in quantitative relationships among the variables.	Manager and O.R. Specialist
↓ Investigate methods for solving the problems as stated above; determine appropriate quantitative tools to be used.	O.R. Specialist
↓ Attempt solutions to the problems; find various solutions; state assumptions underlying these solutions; test alternative solutions.	O.R. Specialist
↓ Determine which solution is most effective because of practical constraints within the organization; decide what the solution means for the organization. ↓	Manager and O.R. Specialist

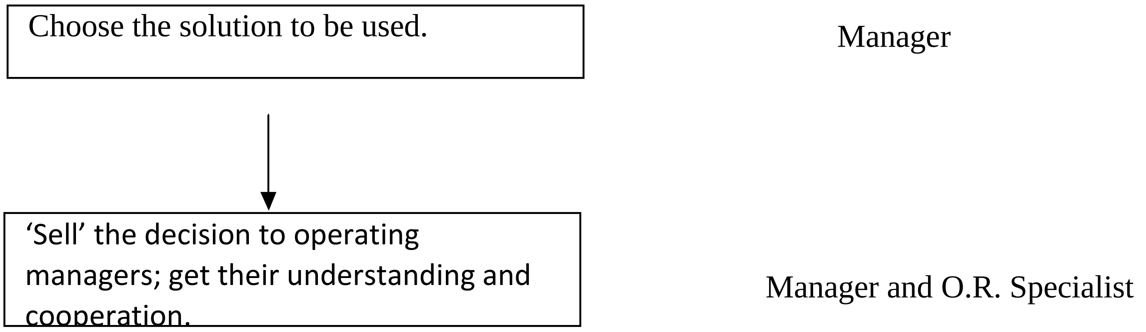

Figure 1-1 Relationship between Manager/Decision Maker and O.R. Specialists

1.5 O.R. Tools and Techniques

Operations Research uses any suitable tools or techniques available. The common frequently used tools/techniques are mathematical procedures, cost analysis, electronic computation. However, operations researchers given special importance to the development and the use of techniques like linear programming, game theory, decision theory, queuing theory, inventory models and simulation. In addition to the above techniques, some other common tools are non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling (PERT/CPM), symbolic Model, information theory, and value theory. There is many other Operations Research tools/techniques also exists. The brief explanations of some of the above techniques/tools are as follows:

Linear Programming:

This is a constrained optimization technique, which optimize some criterion within some constraints. In Linear programming the objective function (profit, loss or return on investment) and constraints are linear. There are different methods available to solve linear programming.

Game Theory:

This is used for making decisions under conflicting situations where there are one or more players/opponents. In this the motive of the players are dichotomized. The success of one player tends to be at the cost of other players and hence they are in conflict.

Decision Theory:

Decision theory is concerned with making decisions under conditions of complete certainty about the future outcomes and under conditions such that we can make some probability about what will happen in future.

Queuing Theory:

This is used in situations where the queue is formed (for example customers waiting for service, aircrafts waiting for landing, jobs waiting for processing in the computer system, etc). The objective here is minimizing the cost of waiting without increasing the cost of servicing.

Inventory Models:

Inventory model make a decisions that minimize total inventory cost. This model successfully reduces the total cost of purchasing, carrying, and out of stock inventory.

Simulation:

Simulation is a procedure that studies a problem by creating a model of the process involved in the problem and then through a series of organized trials and error solutions attempt to determine the best solution. Sometimes this is a difficult/time consuming procedure. Simulation is used when actual experimentation is not feasible or solution of model is not possible.

Non-linear Programming:

This is used when the objective function and the constraints are not linear in nature. Linear relationships may be applied to approximate non-linear constraints but limited to some range, because approximation becomes poorer as the range is extended. Thus, the non-linear programming is used to determine the approximation in which a solution lies and then the solution is obtained using linear methods.

Dynamic Programming:

Dynamic programming is a method of analyzing multistage decision processes. In this each elementary decision depends on those preceding decisions and as well as external factors.

Integer Programming:

If one or more variables of the problem take integral values only then dynamic programming method is used. For example number or motor in an organization, number of passenger in an aircraft, number of generators in a power generating plant, etc.

Markov Process:

Markov process permits to predict changes over time information about the behavior of a system is known. This is used in decision making in situations where the various states are defined. The probability from one state to another state is known and depends on the current state and is independent of how we have arrived at that particular state.

Network Scheduling:

This technique is used extensively to plan, schedule, and monitor large projects (for example computer system installation, R & D design, construction, maintenance, etc.). The aim of this technique is minimize trouble spots (such as delays, interruption, production bottlenecks, etc.) by identifying the critical factors. The different activities and their relationships of the entire project are represented diagrammatically with the help of networks and arrows, which is used for identifying critical activities and path. There are two main types of technique in network scheduling, they are:

Program Evaluation and Review Technique (PERT) – is used when activities time is not known accurately/ only probabilistic estimate of time is available.

Critical Path Method (CPM) – is used when activities time is known accurately.

Information Theory:

This analytical process is transferred from the electrical communication field to O.R. field. The objective of this theory is to evaluate the effectiveness of flow of information with a given system. This is used mainly in communication networks but also has indirect influence in simulating the examination of business organizational structure with a view of enhancing flow of information.

1.6 Applications of Operations Research/ Management Science

Today, almost all fields of business and government utilizing the benefits of Operations Research. There are voluminous of applications of Operations Research. Although it is not feasible to cover all applications of O.R. in brief, the following are the abbreviated set of typical operations research applications to show how widely these techniques are used today:

Accounting:

Assigning audit teams effectively Credit policy analysis

Cash flow planning Developing standard costs

Establishing costs for by products planning of delinquent account strategy

Construction:

Project scheduling, monitoring and control Determination of proper work force Deployment of work force

Allocation of resources to projects

Facilities Planning:

Factory location and size decision Estimation of number of facilities required Hospital planning International logistic system design Transportation loading and unloading Warehouse location decision

Finance:

Building cash management models Allocating capital among various alternatives Building financial planning models Investment analysis

Portfolio analysis Dividend policy making

Manufacturing:

Inventory control

Marketing balance projection Production scheduling Production smoothing

Marketing:

Advertising budget allocation Product introduction timing Selection of Product mix

Deciding most effective packaging alternative

Organizational Behavior / Human Resources:

Personnel planning Recruitment of employees Skill balancing

Training program scheduling

Designing organizational structure more effectively

Purchasing:

Optimal buying Optimal reordering Materials transfer

Research and Development:

R & D Projects control R & D Budget allocation

Planning of Product introduction

1.7 Limitations of Operations Research/ Management Science

Operations Research has number of applications; similarly it also has certain limitations. These limitations are mostly related to the model building and money and time factors problems involved in its application. Some of them are as given below:

a) Distance between O.R. specialist and Manager

   Operations Researchers job needs a mathematician or statistician, who might not be aware of the business problems. Similarly, a manager is unable to understand the complex nature of Operations Research. Thus there is a big gap between the two personnel.

b) Magnitude of Calculations

The aim of the O.R. is to find out optimal solution taking into consideration all the factors. In this modern world these factors are enormous and expressing them in quantitative model and establishing relationships among these require voluminous calculations, which can be handled only by machines.

c) Money and Time Costs

The basic data are subjected to frequent changes, incorporating these changes into the operations research models is very expensive. However, a fairly good solution at present may be more desirable than a perfect operations research solution available in future or after some time.

a) Non-quantifiable Factors

When all the factors related to a problem can be quantifiable only then operations research provides solution otherwise not. The non-quantifiable factors are not incorporated in O.R. models. Importantly O.R. models do not take into account emotional factors or qualitative factors.

b) Implementation

Once the decision has been taken it should be implemented. The implementation of decisions is a delicate task. This task must take into account the complexities of human relations and behavior and in some times only the psychological factors.

1.8 Check Your Progress

There are some activities to check your progress. Answer the followings-

1. Management Science was first used successfully in ____________________
2. The main objective of Management Science models is _________________
3. The main limitation of Management Science models is that they do not include _________
4. When actual experimentation is not feasible, _____________ model is the most preferred one.
5. The first step of Management Science methodology is _____________________

1.9 Summary

Operations Research is relatively a new discipline, which originated in World War II, and became very popular throughout the world. India is one of the few first countries in the world who started using operations research. Operations Research is used successfully not only in military/army operations but also in business, government and industry. Now a day’s operations research is almost used in all the fields.

Proposing a definition to the operations research is a difficult one, because its boundary and content are not fixed. The tools for operations search is provided from the subject’s viz. economics, engineering, mathematics, statistics, psychology, etc., which helps to choose possible alternative courses of action. The operations research tool/techniques include linear programming, non-linear programming, dynamic programming, integer programming, Markov process, queuing theory, etc.

Operations Research has a number of applications. Similarly it has a number of limitations, which is basically related to the time, money, and the problem involves in the model building. Day-by- day operations research is gaining acceptance because it improves decision making effectiveness of the managers. Almost all the areas of business use the operations research for decision making.

1.10 Keywords

OR: Operations Research.

MS: Management Science.

Symbolic Model: An abstract model, generally using mathematical symbols.

Criterion: is measurement, which is used to evaluation of the results.

Integer Programming: is a technique, which ensures only integral values of variables in the problem.

Dynamic Programming: is a technique, which is used to analyze multistage decision process.

Linear Programming: is a technique, which optimizes linear objective function under limited constraints.

Inventory Model: these are the models used to minimize total inventory costs.

Optimization: Means maximization or minimization.

1.11 Self Assessment Questions

Q1. Define Operations Research.

Q2. Describe the relationship between the manager and O.R. specialist. Q3. Explain the various steps in the O.R. development process.

Q4. Discuss the applications of O.R. Q5. Discuss the limitation of O.R.

Q6. Describe different techniques of O.R.

Q7. Discuss few areas of O.R. applications in your organization or organization you are familiar with.

1.12 Answers to Check your Progress:

1. Military operations
2. Optimization
3. Non quantifiable variables
4. Simulation
5. Defining the problem

1.13 References/ Suggested Readings

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

Sharma, J.K., 1989. Mathematical Models in Operations Research, Tata McGraw Hill Publishing Company Ltd., New Delhi.

Beer, Stafford, 1966. Decision and Control, John Wiley & Sons, Inc., New York.

Levin, Rubin, Stinson, Gardner, 1992. Quantitative Approaches to Management, Tata McGraw Hill Publishing Company Ltd. New Delhi.

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.: 2	Vetter:
Linear Programming Problem (LPP): Graphical Method	

Structure

1.1 Introduction to Linear Programming

1.2 Linear Programming Problem Formulation

1.3 Formulation with Different Types of Constraints

1.4 Graphical Analysis of Linear Programming

1.5 Graphical Linear Programming Solution

1.6 Multiple Optimal Solutions

1.7 Unbounded Solution

1.8 Infeasible Solution

1.9 Check your progress

1.10 Summary

1.11 Keywords

1.12 Self Assessment Test

1.13 Answers to check your progress

1.14 References/ Suggested Readings

Learning Objectives

After Studying this lesson, students will be able to:

- ❖ Formulate Linear Programming Problem
- ❖ Identify the characteristics of linear programming problem
- ❖ Make a graphical analysis of the linear programming problem
- ❖ Solve the problem graphically
- ❖ Identify the various types of solutions

1.1 Introduction to Linear Programming

Linear Programming is a special and versatile technique which can be applied to a variety of management problems viz. Advertising, Distribution, Investment, Production, Refinery Operations, and Transportation analysis. The linear programming is useful not only in industry and business but also in non-profit sectors such as Education, Government, Hospital, and Libraries. The linear programming method is applicable in problems characterized by the presence of decision variables. The objective function and the constraints can be expressed as **linear functions** of the **decision variables**. The decision variables represent quantities that are, in some sense, controllable inputs to the system being modeled.

An objective function represents some principal objective criterion or goal that measures the effectiveness of the system such as maximizing profits or productivity, or minimizing cost or consumption. There is always some practical limitation on the availability of resources viz. man, material, machine, or time for the system. These constraints are expressed as linear equations involving the decision variables. Solving a linear programming problem means determining actual values of the decision variables that optimize the objective function subject to the limitation imposed by the constraints.

The main important feature of linear programming model is the presence of linearity in the problem. The use of linear programming model arises in a wide variety of applications. Some model may not be strictly linear, but can be made linear by applying appropriate mathematical transformations. Still some applications are not at all linear, but can be effectively approximated by linear models. The ease with which linear programming models can usually be solved makes an attractive means of dealing with otherwise intractable nonlinear models.

1.2 Linear Programming Problem Formulation

The linear programming problem formulation is illustrated through a product mix problem. The product mix problem occurs in an industry where it is possible to manufacture a variety of products. A product has a certain margin of profit per unit, and uses a common pool of limited resources. In this case the linear programming technique identifies the products combination which will maximize the profit subject to the availability of limited resource constraints.

Example 2.1:

Suppose an industry is manufacturing two types of products P1 and P2. The profits per Kg of the two products are Rs.30 and Rs.40 respectively. These two products require processing in three types of machines. The following table shows the available machine hours per day and the time required on each machine to produce one Kg of P1 and P2. Formulate the problem in the form of linear programming model.

Profit/Kg	P1 Rs.30	P2 Rs.40	Total available Machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

Solution:

The procedure for linear programming problem formulation is as

follows: Introduce the decision variable as follows:

Let x_1 = amount of P1

x_2 = amount of P2

In order to maximize profits, we establish the

objective function as $30x_1 + 40x_2$

Since one Kg of P1 requires 3 hours of processing time in machine 1 while the corresponding requirement of P2 is 2 hours. So, the first constraint can be expressed as

$3x_1 + 2x_2 \leq 600$

Similarly, corresponding to machine 2 and 3 the constraints are

$3x_1 + 5x_2 \leq 800$

$5x_1 + 6x_2 \leq 1100$

In addition to the above there is no negative production, which may be represented

algebraically as $x_1 \geq 0$; $x_2 \geq 0$

Thus, the product mix problem in the linear programming model is as follows:

Maximize

$30x_1 + 40x_2$

Subject to:

$3x_1 + 2x_2 \leq 600$

$3x_1 + 5x_2 \leq 800$

$5x_1 + 6x_2 \leq 1100$

$x_1 \geq 0, x_2 \geq 0$

1.3 Formulation with Different Types of Constraints

The constraints in the previous example 2.1 are of “less than or equal to” type. In this section we are going to discuss the linear programming problem with different constraints, which is illustrated in the following Example 2.2.

Example 2.2:

A company owns two flour mills namely A and B, which have different production capacities for high, medium and low quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs.2000 and Rs.1500 per day to run mill A and B respectively. On a day, Mill A produces 6, 2 and 4 quintals of high, medium and low quality flour, Mill B produces 2, 4 and 12 quintals of high, medium and low quality flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically.

Solution:

Let us define x1 and x2 are the mills A and B. Here the objective is to minimize the cost of the machine runs and to satisfy the contract order. The linear programming problem is given by

Minimize

$$2000x_1 + 1500x_2$$

Subject to:

$$6x_1 + 2x_2 \geq 8$$

$$2x_1 + 4x_2 \geq 12$$

$$4x_1 + 12x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0$$

1.4 Graphical Analysis of Linear Programming

This section shows how a two-variable linear programming problem is solved graphically, which is illustrated as follows:

Example 2.3:

Consider the product mix problem discussed in section 2.2

Maximize

$$30x_1 + 40x_2$$

Subject to:

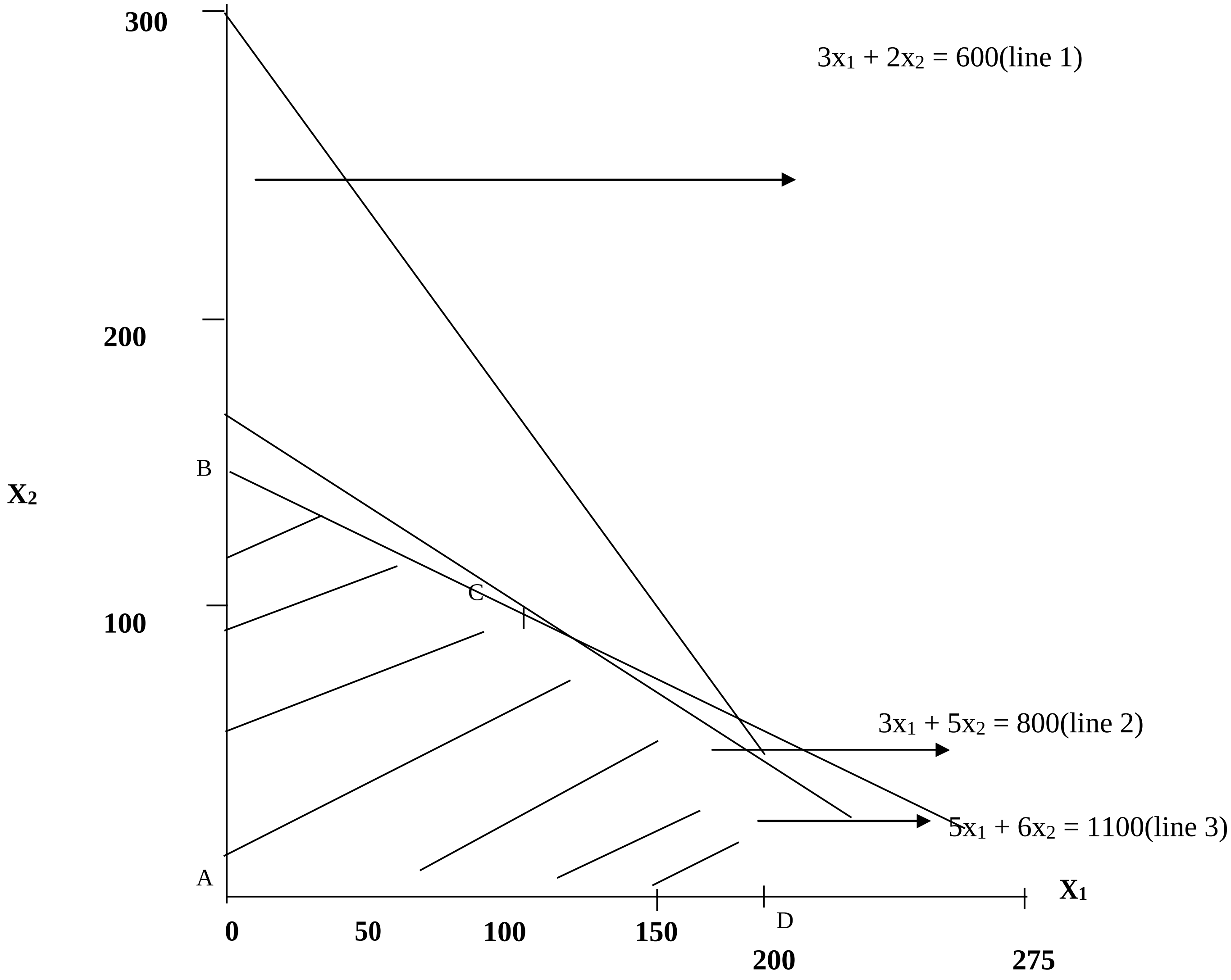
$$3x_1 + 2x_2 \leq 600$$

$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

$$x_1 \geq 0, x_2 \geq 0$$

From the first constraints $3x_1 + 2x_2 \leq 600$, draw the line $3x_1 + 2x_2 = 600$ which passes through the point (200, 0) and (0, 300). This is shown in the following graph as line 1.



Graph 1: Three closed half planes and Feasible Region

Half Plane - A linear inequality in two variables is called as a half plane.

Boundary - The corresponding equality (line) is called as the boundary of the half plane. Close Half Plane – Half plane with its boundary is called as a closed half plane.

In this case we must decide in which side of the line $3x_1 + 2x_2 = 600$ the half plane is located. The easiest way to solve the inequality for x_2 is $3x_1 \leq 600 - 2x_2$

And for the fixed x_1, the coordinates satisfy this inequality are smaller than the corresponding ordinate on the line and thus the inequality is satisfied for all the points below the line 1.

Similarly, we have to determine the closed half planes for the inequalities $3x_1 + 5x_2 \leq 800$ and $5x_1 + 6x_2 \leq 1100$ (line 2 and line 3 in the graph). Since all the three constraints must be satisfied simultaneously we have consider the intersection of these three closed half planes. The complete intersection of these three closed half planes is shown in the above graph as ABCD. The region ABCD is called the feasible region, which is shaded in the graph.

Feasible Solution:

Any non-negative value of x_1, x_2 that is $x_1 \geq 0$ and $x_2 \geq 0$ is known as feasible solution of the linear programming problem if it satisfies all the existing constraints.

Feasible Region:

The collection of all the feasible solution is called as the feasible region.

Example 2.4:

In the previous example we discussed about the less than or equal to type of linear programming problem, i.e. maximization problem. Now consider a minimization (i.e. greater than or equal to type) linear programming problem formulated in Example 2.2.

Minimize

$$2000x_1 + 1500x_2$$

Subject to:

$$6x_1 + 2x_2 \geq 8$$

$$2x_1 + 4x_2 \geq 12$$

$$4x_1 + 12x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0$$

The three lines $6x_1 + 2x_2 = 8$, $2x_1 + 4x_2 = 12$, and $4x_1 + 12x_2 = 24$ passes through the point (1.3,0) (0,4), (6,0) (0,3) and (6,0) (0,2). The feasible region for this problem is shown in the following Graph 2. In this problem the constraints are of greater than or equal to type of feasible region, which is bounded on one side only.

In this section we are going to describe linear programming graphical solution for both the maximization and minimization problems, discussed in Example 2.3 and Example 2.4.

Example 2.5:

Consider the maximization problem described in Example 2.3.

Maximize

$$30x_1 + 40x_2$$

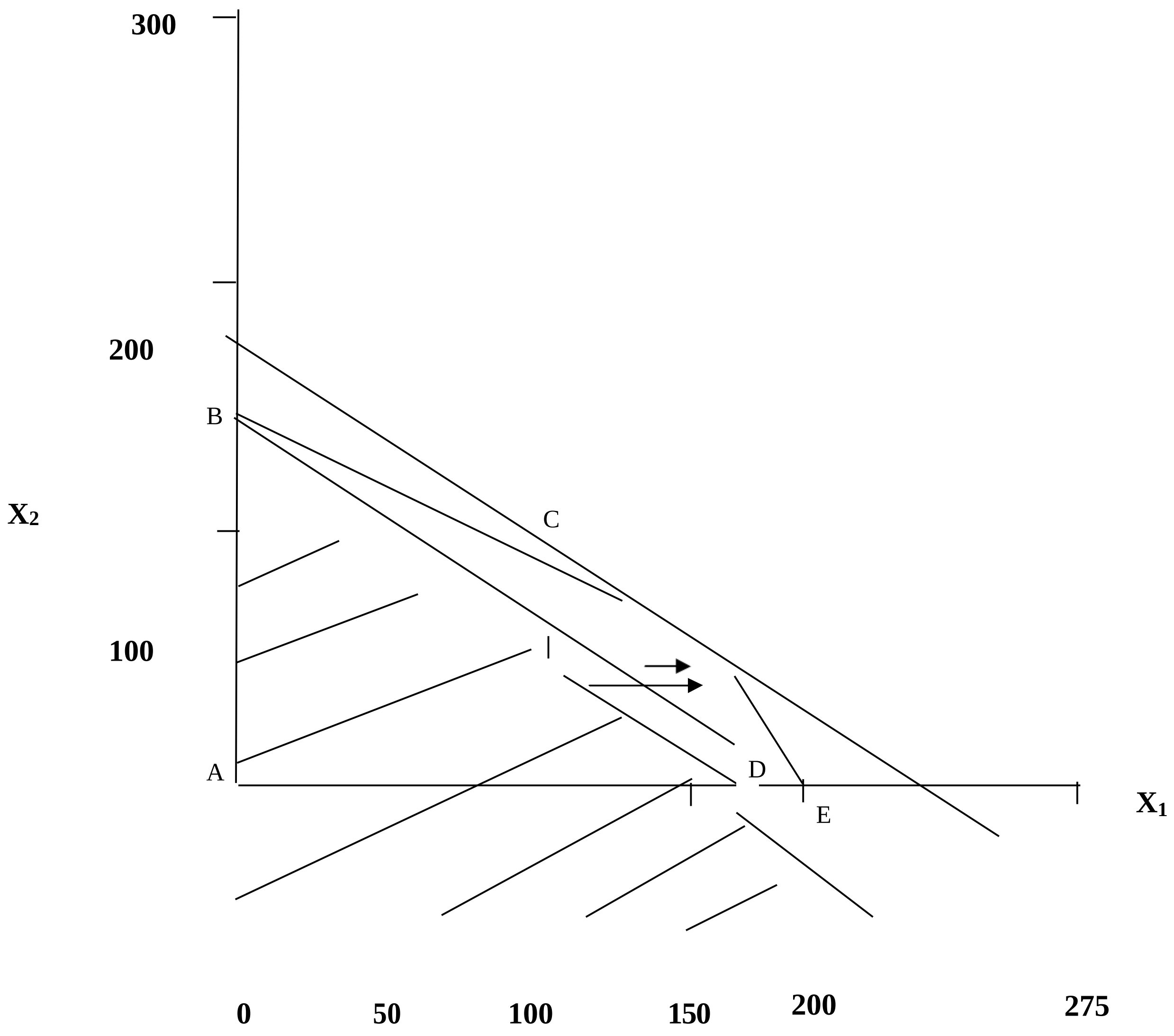
Subject to:

$$3x_1 + 2x_2 \leq 600$$
$$3x_1 + 5x_2 \leq 800$$
$$5x_1 + 6x_2 \leq 1100$$
$$x_1 \geq 0,\ x_2 \geq 0$$

$M = 30x_1 + 40x_2$

The feasible region identified in the Example 2.3 is a convex polygon, which is illustrated in the following Graph 3. The extreme point of this convex region are A, B, C, D and E.

Graph 3: Graphical Linear Programming Solution



In this problem the objective function is $30x_1 + 40x_2$. Let be M is a parameter, the graph $30x_1 + 40x_2 = M$ is a group of parallel lines with slope – 30/40. Some of these lines intersects the feasible region and contains many feasible solutions, whereas the other lines miss and contain no feasible solution. In order to maximize the objective function, we find the line of this family that intersects the feasible region and is farthest out from the origin. Note that the farthest is the line from the origin the greater will be the value of M.

Observe that the line $30x_1 + 40x_2 = M$ passes through the point D, which is the intersection of the lines $3x_1 + 5x_2 = 800$ and $5x_1 + 6x_2 = 1100$ and has the coordinates $x_1 = 170$ and $x_2 = 40$. Since D is the only feasible solution on this line the solution is unique. The value of M is 6700, which is the objective function maximum value. The optimum value variables are $x_1 = 170$ and $X_2 = 40$.

The following Table 1 shows the calculation of maximum value of the objective function.

Extreme Point	Coordinates X_1	X2	Objective Function $30x_1 + 40x_2$
A	$X_1 = 0$	$X_2 = 0$	0
B	$X_1 = 0$	$X_2 = 160$	6400
C	$X_1 = 110$	$X_2 = 70$	6100
D	$X_1 = 170$	$X_2 = 40$	6700
E	$X_1 = 200$	$X_2 = 0$	6000

Table 1: Shows the objective function Maximum value calculation

Example 2.6:

Consider the minimization problem described in Example 2.4.

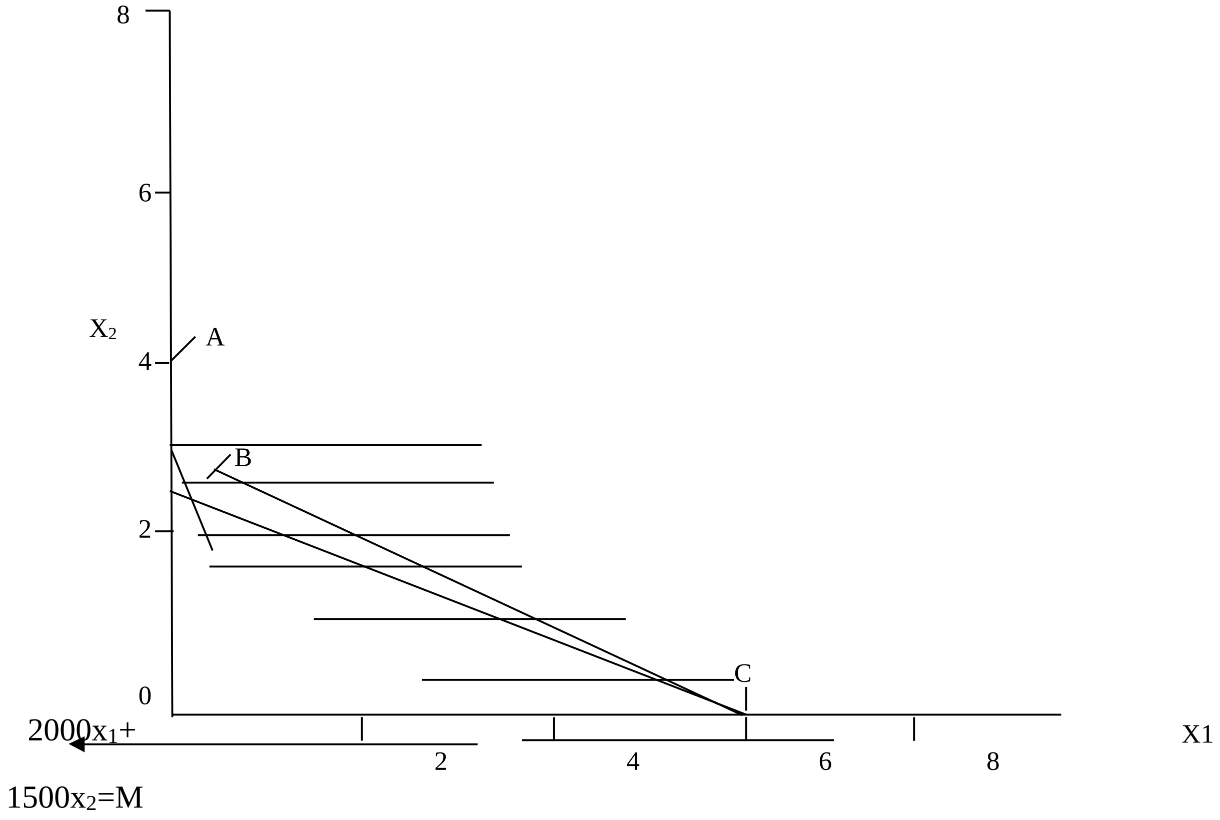
Minimize

$$2000x_1 + 1500x_2$$

Subject to:

$$6x_1 + 2x_2 \geq 8$$
$$2x_1 + 4x_2 \geq 12$$
$$4x_1 + 12x_2 \geq 24$$
$$x_1 \geq 0,\ x_2 \geq 0$$

The feasible region for this problem is illustrated in the following Graph 4. Here each of the half planes lies above its boundary. In this case the feasible region is infinite. In this case, we are concerned with the minimization; also it is not possible to determine the maximum value. As in the previous example let us introduce a parameter M in the objective function i.e. $2000x_1 + 1500x_2 = M$ and draw the lines for different values of M, which is shown in the following Table 2.



Graph 4: Graphical Linear Programming Solution

Extreme Point	Coordinates X_1	X_2	Objective Function $2000x_1 + 1500x_2$
A	$X_1 = 0$	$X_2 = 4$	6000
B	$X_1 = 0.5$	$X_2 = 2.75$	5125
C	$X_1 = 6$	$X_2 = 0$	12000

Table 2: Shows the objective function Minimum value computation

The minimum value is 5125 at the extreme point B, which is the value of the M (objective function). The optimum values variables are $X_1 = 0.5$ and $X_2 = 2.75$.

1.6 Multiple Optimal Solutions

When the objective function passed through only the extreme point located at the intersection of two half planes, then the linear programming problem possess unique solutions. The previous examples i.e. Example 2.5 and Example 2.6 are of this types (which possessed unique solutions).

When the objective function coincides with one of the half planes generated by the constraints in the problem, will possess multiple optimal solutions. In this section we are going to discuss about the multiple optimal solutions of linear programming problem with the help of the following Example 2.7.

Example 2.7:

A company purchasing scrap material has two types of scarp materials available. The first type has 30% of material X, 20% of material Y and 50% of material Z by weight. The second type has 40% of material X, 10% of material Y and 30% of material Z. The costs of the two scraps are Rs.120 and Rs.160 per kg respectively. The company requires at least 240 kg of material X, 100 kg of material Y and 290 kg of material Z. Find the optimum quantities of the two scraps to be purchased so that the company requirements of the three materials are satisfied at a minimum cost.

Solution

First we have to formulate the linear programming model. Let us introduce the decision variables x1 and x2 denoting the amount of scrap material to be purchased. Here the objective is to minimize the purchasing cost. So, the objective function here is

Minimize

$$120x_1 + 160x_2$$

Subject to:

$0.3x_1 + 0.4x_2 \geq 240$

$0.2x_1 + 0.1x_2 \geq 100$

$0.5x_1 + 0.3x_2 \geq 290$

$x_1 \geq 0; x_2 \geq 0$

Multiply by 10 both sides of the inequalities, then the problem becomes Minimize

$120x_1 + 160x_2$

Subject to:

$3x_1 + 4x_2 \geq 2400$

$2x_1 + x_2 \geq 1000$

$5x_1 + 3x_2 \geq 2900$

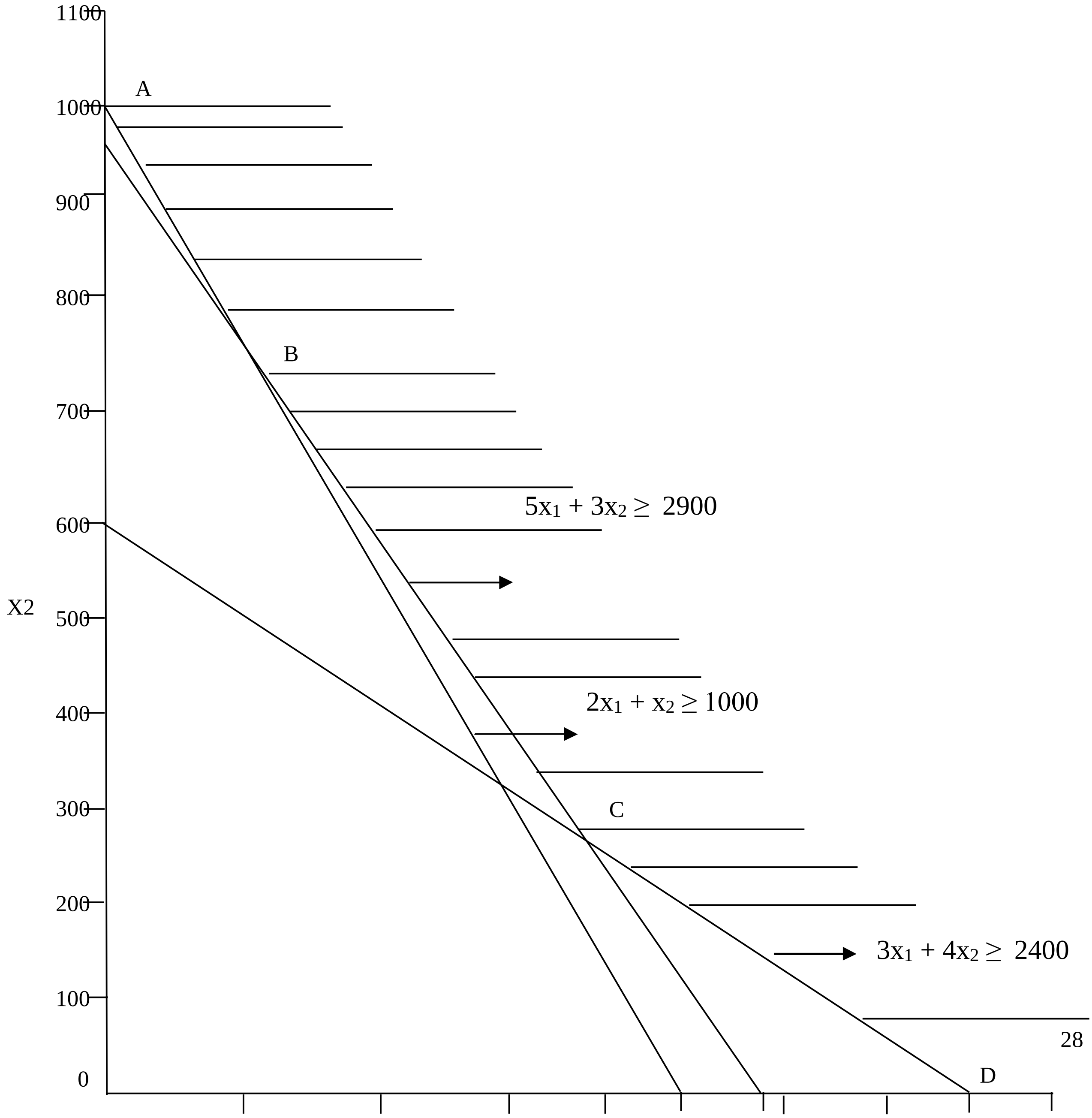
$x_1 \geq 0; x_2 \geq 0$

Let us introduce parameter M in the objective function i.e. $120x_1 + 160x_2 = M$. Then we have to determine the different values for M, which is shown in the following Table 3.

Extreme Point	Coordinates X_1	X_2	Objective Function $120x_1 + 160x_2$
A	$X_1 = 0$	$X_2 = 1000$	160000
B	$X_1 = 150$	$X_2 = 740$	136400
C	$X_1 = 400$	$X_2 = 300$	96000
D	$X_1=800$	$X_2=0$	96000

Table 3: Shows the calculation of Minimum objective function value

Note that there are two minimum value for the objective function (M=96000). The feasible region and the multiple solutions are indicated in the following Graph 5.



Graph 5: Feasible Region, Multiple Optimal Solutions

The extreme points are A, B, C, and D. One of the objective functions $120x_1 + 160x_2 = M$ family coincides with the line CD at the point C with value M=96000, and the optimum value variables are $x_1 = 400$, and $x_2 = 300$. And at the point D with value M=96000, and the optimum value variables are $x_1 = 800$, and $x_2 = 0$.

Thus, every point on the line CD minimizes objective function value and the problem contains multiple optimal solutions.

1.7 Unbounded Solution

When the feasible region is unbounded, a maximization problem may don't have optimal solution, since the values of the decision variables may be increased arbitrarily. This is illustrated with the help of the following problem.

Maximize

$$3x_1 + x_2$$

Subject to:

$$x_1 + x_2 \geq 6$$

$$-x_1 + x_2 \leq 6$$

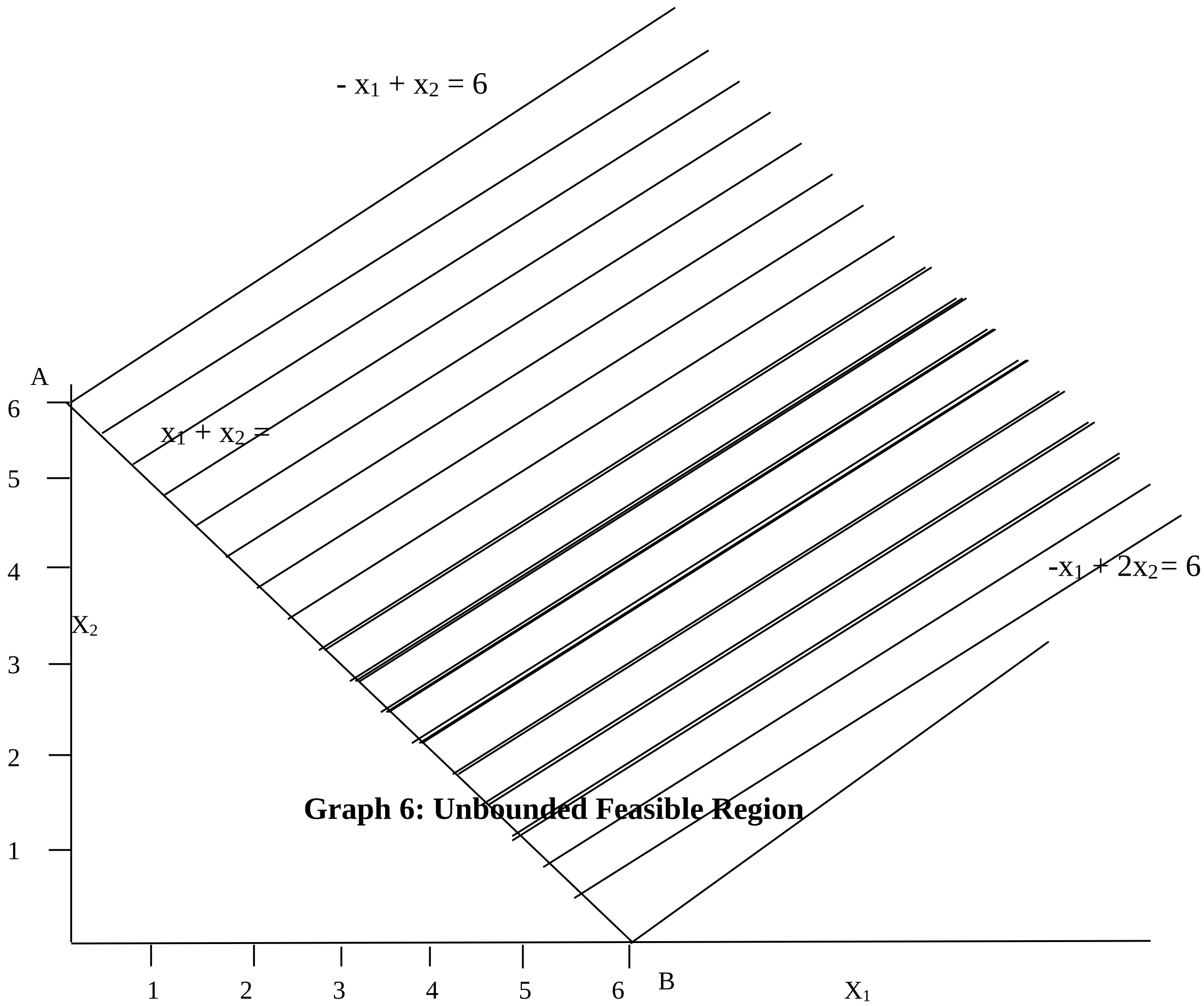
$$-x_1 + 2x_2 \geq -6$$

and

$$x_1, x_2 \geq 0$$

Graph 6 shows the unbounded feasible region and demonstrates that the objective function can be made arbitrarily large by increasing the values of x_1 and x_2 within the unbounded feasible region. In this case, there is no point (x_1, x_2) is optimal because there are always other feasible points for which objective function is larger. Note that it is not the unbounded feasible region alone that precludes an optimal solution. The minimization of the function subject to the constraints shown in the Graph 6 would be solved at one the extreme point (A or B).

The unbounded solutions typically arise because some real constraints, which represent a practical resource limitation, have been missed from the linear programming formulation. In such situation the problem needs to be reformulated and re-solved.



Graph 6: Unbounded Feasible Region

1.8 Infeasible Solution

A linear programming problem is said to be infeasible if no feasible solution of the problem exists. This section describes infeasible solution of the linear programming problem with the help of the following Example 2.8.

Example 2.8:

Minimize

$200x_1 + 300x_2$

Subject to:

$0.4x_1 + 0.6x_2 \geq 240$

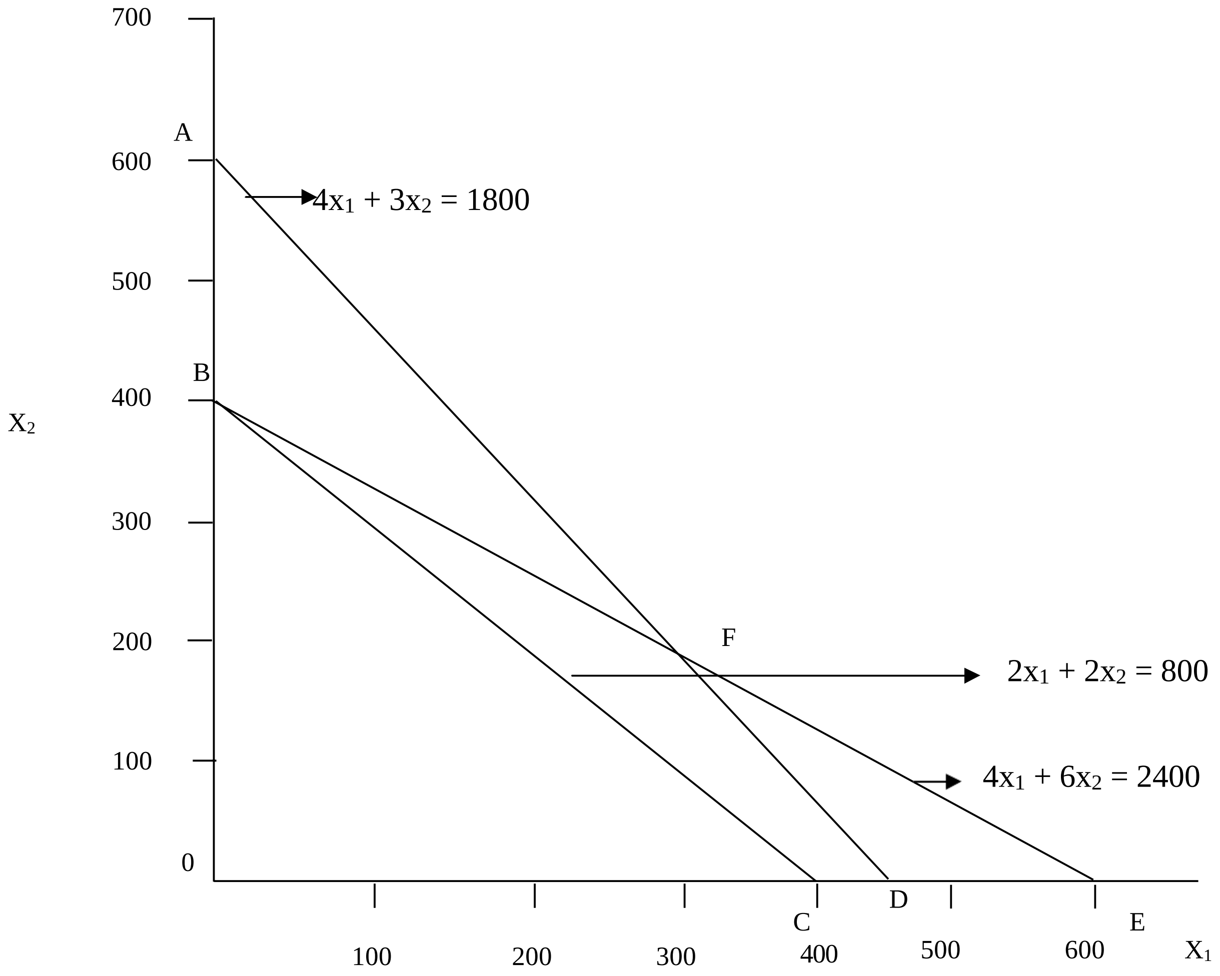
$0.2x_1 + 0.2x_2 \leq 80$

$0.4x_1 + 0.3x_2 \geq 180$

$x_1, x_2 \geq 0$

On multiplying both sides of the inequalities by 10, we get

$$4x_1 + 6x_2 \geq 2400$$
$$2x_1 + 2x_2 \leq 800$$
$$4x_1 + 3x_2 \geq 1800$$



Graph 7: Infeasible Solution

The region right of the boundary AFE includes all the solutions which satisfy the first ($4x_1 + 6x_2 \geq 2400$) and the third ($4x_1 + 3x_2 \geq 1800$) constraints. The region left of the BC contains all solutions which satisfy the second constraint ($2x_1 + 2x_2 \leq 800$)

Hence, there is no solution satisfying all the three constraints (first, second, and third). Thus, the linear problem is infeasible. This is illustrated in the above Graph 7

.

1.9 Check Your Progress

There are some activities to check your progress. Answer the followings

1. In a linear programming problem with maximization objective, the feasible region would be ___________
2. In a linear programming problem with minimization objective, the feasible region would be ___________
3. If a linear programming problem does not have a feasible region, it means that problem________________
4. Some constraints do not affect objective function are known as ____________
5. When optimal values are put in any constraint and the right and left hand sides are equal, this constraint will be known as _______________
6. A linear programming problem may have ________________solutions.

1.10 Summary

In Operations Research linear programming is a versatile technique with wide applications in various management problems. Linear Programming problem has a number of characteristics. That is first we have to identify the decision variable. The problem must have a well defined objective function, which are expressed in terms of the decision variables.

The objective function may have to be maximized when it indicates the profit or production or contribution. If the objective function represents cost, in this case the objective function has to be minimized.

The management problem is expressed in terms of the decision variables with the objective function and constraints. A linear programming problem is solved graphically if it contains only two variables.

1.11 Keywords

Objective Function: is a linear function of the decision variables representing the objective of the manager/decision maker.

Constraints: are the linear equations or inequalities arising out of practical limitations.

Decision Variables: are some physical quantities whose values indicate the solution.

Feasible Solution: is a solution which satisfies all the constraints (including the non-negative) presents in the problem.

Feasible Region: is the collection of feasible solutions.

Multiple Solutions: are solutions each of which maximize or minimize the objective function.

Unbounded Solution: is a solution whose objective function is infinite.

Infeasible Solution: means no feasible solution.

1.12 Self Assessment Test

1) A juice company has its products viz. canned apple and bottled juice with profit margin Rs.4 and Rs.2 respectively pre unit. The following table shows the labour, equipment, and ingredients to produce each product per unit.

	Canned Apple	Bottled Juice	Total
Labour	2.0	3.0	12.0
Equipment	3.2	1.0	8.0
Ingredients	2.4	2.0	9.0

Formulate the linear programming problem (model) specifying the product mix which will maximize the profit without exceeding the levels of resources.

2) An organization is interested in the analysis of two products which can be produces from the idle time of labour, machine and investment. It was notified on investigation that the labour requirement of the first and the second products was 4 and 5 units respectively and the total available man hours was 48. Only first product required machine hour utilization of one hour per unit and at present only 10 spare machine hours are available. Second product needs one unit of byproduct per unit and the daily availability of the byproduct is 12 units. According to the marketing department the sales potential of first product cannot exceed 7 units. In a competitive market, first product can be sold at a profit of Rs.6 and the second product at a profit of Rs.10 per unit.

Formulate the problem as a linear programming model. Also determine graphically the feasible region. Identify the redundant constraints if any.

3) Find graphically the feasible region of the linear programming problem given in Q1.

4) A bed mart company is in the business of manufacturing beds and pillows. The company has 40 hours for assembly and 32 hours for finishing work per day. Manufacturing of a bed requires 4 hours for assembly and 2 hours in finishing. Similarly a pillow requires 2 hours for assembly and 4 hours for finishing. Profitability

analysis indicates that every bed would contribute Rs.80, while a pillow contribution is Rs.55 respectively. Find out the daily production of the company to maximize the contribution (profit).

Q5. Maximize

$1170x_1 + 1110x_2$

Subject to:

$9x_1 + 5x_2 \geq 500$

$7x_1 + 9x_2 \geq 300$

$5x_1 + 3x_2 \leq 1500$

$7x_1 + 9x_2 \leq 1900$

$2x_1 + 4x_2 \leq 1000$

$x_1, x_2 \geq 0$

Find graphically the feasible region and the optimal solution.

Q6. Solve the following LP problem

graphically Minimize

$2x_1 + 1.7x_2$

Subject to:

$0.15x_1 + 0.10x2 \geq 1.0$

$0.75x_1 + 1.70x_2 \geq 7.5$

$1.30x_1 + 1.10x_2 \geq 10.0$

$x_1, x_2 \geq 0$

Q7. Solve the following LP problem graphically

Maximize

$2x_1 + 3x_2$

Subject to:

$x_1 - x_2$
$\leq 1 \quad x_1$
$+ \; x_2 \; \geq$
$3 \quad x_1,$
$x_2 \geq 0$

Q8. Graphically solve the following problem of LLP

Maximize

$3x_1 + 2x_2$

Subject to:

$2x_1 - 3x_2 \geq 0$

$3x_1 + 4x_2 \leq -12$

$x_1, x_2 \geq 0$

Q9. Solve the following problem graphically

Maximize
$4x_1 + 4x_2$

Subject to:

$-2x_1 + x_2 \leq 1$

$x_1 \leq 2$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

1.13 Answers to Check Your Progress:

1. Bounded
2. Unbounded
3. Does not have any solution
4. Redundant Constraints
5. Binding Constraint
6. Multiple optimum solutions

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.: 3	Vetter:
Linear Programming Problem (LPP): Simplex Method	

Structure:

1.1 Introduction

1.2 Basics of Simplex Method

1.3 Simplex Method Computation

1.4 Simplex Method with More than Two Variables

1.5 Two Phase and M Method

1.6 Two Phase Method

1.7 M Method

1.8 Multiple Solutions

1.9 Unbounded Solution

1.10 Infeasible Solution

1.11 Check your progress

1.12 Summary

1.13 Keywords

1.14 Self Assessment Test

1.15 Answers to check your progress

1.16 References/ Suggested Readings

Learning Objectives

After Studying this lesson, students will be able to:

- ❖ Understand the basics of simplex method
- ❖ Explain the simplex calculations
- ❖ Describe various solutions of Simplex Method
- ❖ Understand two phase and M method

1.1 Introduction

The Linear Programming with two variables can be solved graphically. The graphical method of solving linear programming problem is of limited application in the business problems as the number of variables is substantially large. If the linear programming problem has larger number of variables, the suitable method for solving is Simplex Method. The simplex method is an iterative process, through which it reaches ultimately to the minimum or maximum value of the objective function.

The simplex method also helps the decision maker/manager to identify the following:

- ✓ Redundant Constraints
- ✓ Multiple Solutions
- ✓ Unbounded Solution
- ✓ Infeasible Problem

1.2 Basics of Simplex Method

The basic of simplex method is explained with the following linear programming problem.

Example:

Maximize
$60x_1 + 70x_2$

Subject to:
$2x_1 + x_2 \leq 300$

$3x_1 + 4x_2 \leq 509$

$4x_1 + 7x_2 \leq 812$

$x_1, x_2 \geq 0$

Solution
First we introduce the variables
$s_3, s_4, s_5 \geq 0$

So that the constraints becomes equations, thus

$2x_1 + x_2 + s_3 = 300$

$3x_1 + 4x_2 + s_4 = 509$

$4x_1 + 7x_2 + s_5 = 812$

Corresponding to the three constraints, the variables s3, s4, s5 are called as slack variables. Now, the system of equation has three equations and five variables.

There are two types of solutions they are basic and basic feasible, which are discussed as follows:

Basic Solution

We may equate any two variables to zero in the above system of equations, and then the system will have three variables. Thus, if this system of three equations with three variables is solvable such a solution is called as basic solution.

For example suppose we take x_1=0 and x_2=0, the solution of the system with remaining three variables is s_3=300, s_4=509 and s_5=812, this is a basic solution and the variables s_3, s_4, and s_5 are known as basic variables where as the variables x_1, x_2 are known as non-basic variables.

The number of basic solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is m and the number of variables including the slack variables is n then there are at most ${}^{n}C_{n-m} = {}^{n}C_{m}$ basic solutions.

Basic Feasible Solution

A basic solution of a linear programming problem is called as basic feasible solutions if it is feasible it means all the variables are non-negative. The solution s_3=300, s_4=509 and s_5=812 is a basic feasible solution.

The number of basic feasible solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is m and the number of variables including the slack variables is n then there are at most ${}^{n}C_{n-m} = {}^{n}C_{m}$ basic feasible solutions.

Every basic feasible solution is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of given constraints. It is impossible to identify the extreme points geometrically if the problem has several variables but the extreme points can be identified using basic feasible solutions. Since one the basic feasible solution will maximize or minimize the objective function, the searching of extreme points can be carry out starting from one basic feasible solution to another.

The Simplex Method provides a systematic search so that the objective function increases in the cases of maximization progressively until the basic feasible solution has been identified where the objective function is maximized.

1.3 Simplex Method Computation

This section describes the computational aspect of simplex method. Consider the following linear programming problem

Maximize

$60x_1 + 70x_2$

Subject to:

$2x_1 + x_2 + s_3 = 300$

$3x_1 + 4x_2 + s_4 = 509$

$4x_1 + 7x_2 + s_5 = 812$

$x_1, x_2, s_3, s_4, s_5 \geq 0$

The profit $Z = 60x_1 + 70x_2$ i.e. Maximize $60x_1 + 70x_2$. The standard form can be summarized in a compact table form as shown in table-1

In this problem the slack variables s3, s4, and s5 provide a basic feasible solution from which the simplex computation starts. That is s_3==300, s_4=509 and s_5=812. This result follows because of the special structure of the columns associated with the slacks.

If z represents profit then z = 0 corresponding to this basic feasible solution. We represent by CB the coefficient of the basic variables in the objective function and by XB the numerical values of the basic variable.

So that the numerical values of the basic variables are: XB1=300, XB2=509, XB3=812. The profit $z = 60x_1+70x_2$ can also expressed as $z- 60x_1-70x_2=0$. The simplex computation starts with the first compact standard simplex table as given below:

Table-1

C_B	Basic Variables	C_j XB	60 x_1	70 x_2	0 s_3	0 s_4	0 s_5
0	s_3	300	2	1	1	0	0
0	s_4	509	3	4	0	1	0
0	s_5	812	4	7	0	0	1
	Z		-60	-70	0	0	0

In the objective function the coefficients of the variables are CB1=CB2=CB3=0. The top most row of the Table 1 denotes the coefficient of the variables x_1, x_2, s_3, s_4, s_5 of the objective function respectively. The column under x1 indicates the coefficient of x_1 in the three equations respectively. Similarly the remaining column also formed.

On seeing the equation $z = 60x_1+70x_2$ we may observe that if either x_1 or x_2, which is currently non-basic is included as a basic variable so that the profit will increase. Since the coefficient of x_2 is higher we choose x_2 to be included as a basic variable in the next iteration. An equivalent criterion of choosing a new basic variable can be obtained the last row of Table 1 i.e. corresponding to z.

Since the entry corresponding to x_2 is smaller between the two negative values, x_2 will be included as a basic variable in the next iteration. However with three constraints there can be only three basic variables.

Thus, by bringing x_2 a basic variable one of the existing basic variables becomes non-basic. The question here is How to identify this variable? The following statements give the solution to this question.

Consider the first equation i.e. $2x_1 + x_2 + s_3 = 300$

From this equation

$2x_1+s_3 = 300-x_2$

But, $x_1= 0$. Hence, in order that $s_3 \geq 0$

$300 - x_2 \geq 0$

i.e. $x_2 \leq 300$

Similarly consider the second equation i.e. $3x_1 + 4x_2+ s_4= 509$

From this equation

$3x_1+s_4 = 509 - 4x_2$

But, $x_1 = 0$. Hence, in order that $s_4 \geq 0$

$509 - 4x_2 \geq 0$

i.e. $x_2 \leq 509/9$

Similarly consider the third equation i.e. $4x_1 + 7x_2 + s_5 = 812$; From this equation

$4x_1+s_5 = 812 - 7x_2$

But, $x_1 = 0$. Hence, in order that $s_5 \geq 0$

$812 - 7x_2 \geq 0$

i.e. $x_2 \leq 812/7$ Therefore the three equation lead to

$x_2 \leq 300$, $x_2 \leq 509/9$, $x_2 \leq 812/7$

Thus x_2 = Min ($x_2 \leq 300$, $x_2 \leq 509/9$, $x_2 \leq 812/7$) it means x_2=Min ($x_2 \leq 300/1$, $x_2 \leq 509/9$, $x_2 \leq 812/7$) =116

Therefore $x_2 = 116$

If $x_2 = 116$, you may be note from the third equation $7x_2+s_5=812$

i.e. $s_5 = 0$

Thus, the variable s_5 becomes non-basic in the next iteration. So that the revised values of the other two basic variables are

$S_3 = 300 - x_2 = 184$

$S_4=509-4*116=45$

Refer to Table 1, we obtain the elements of the next Table i.e. Table 2 using the following rules:

1. We allocate the quantities which are negative in the z-row. Suppose if all the quantities are positive, the inclusion of any non-basic variable will not increase the value of the objective function. Hence the present solution maximizes the objective function. If there are more than one negative values we choose the variable as a basic variable corresponding to which the z value is least as this is likely to increase the more profit.
2. Let x_j be the incoming basic variable and the corresponding elements of the j^{th} row column be denoted by Y_{1j}, Y_{2j} and Y_{3j} respectively. If the present values of the basic variables are XB1, XB2 and XB3 respectively, then we can compute.

   Min [XB1/Y_{1j}, XB2/Y_{2j}, XB3/Y_{3j}] for Y_{1j}, Y_{2j}, $Y_{3j} > 0$.

   Note that if any $Yij \leq 0$, this need not be included in the comparison. If the minimum occurs corresponding to XBr/Yrj then the r^{th} basic variable will become non-basic in the next iteration.
3. Using the following rules the Table 2 is computed from the Table 1.

   - The revised basic variables are s_3, s_4 and x_2. Accordingly, we make CB1=0, CB2=0 and CB3=70.
   - As x2 is the incoming basic variable we make the coefficient of x_2 one by dividing each element of row-3 by 7. Thus the numerical value of the element corresponding to x_1 is 4/7, corresponding to s_5 is 1/7 in Table 2.

- The incoming basic variable should appear only in the third row. So we multiply the Third - row of Table 2 by 1 and subtract it from the first-row of Table 1 element by element. Thus the element corresponding to x_2 in the first-row of Table 2 is 0.

Therefore the element corresponding to x_1 is
2-1*4/7 = 10/7 and the element corresponding to s5 is 0 -1* 1/7 = -1/7

In this way we obtain the elements of the first and the second row in Table 2. In Table 2 the numerical values can also be calculated in a similar way.

Table-2

C_B	Basic Variables	C_j / X_B	60 / x_1	70 / x_2	0 / s_3	0 / s_4	0 / s_5
0	s_3	184	10/7	0	1	0	-1/7
0	s_4	45	5/7	0	0	1	-4/7
70	x_2	116	4/7	1	0	0	1/7
	Zj – Cj		-140/7	0	0	0	70/7

Let CB1, CB2, Cb3 be the coefficients of the basic variables in the objective function. For example in Table 2 CB1=0, CB2=0 and CB3=70. Suppose corresponding to a variable J, the quantity Zj is defined as Zj = CB1, Y1+CB2, Y2j+CB3Y3j. Then the z-row can also be represented as Zj - Cj.

For example:
$z_1 - c_1$ = 10/7 * 0 + 5/7 * 0 + 70 * 4/7 - 60 = -140/7 $z_5 - c_5$ = -1/7 * 0 - 4/7 *0 + 1/7*70-0 = 70/7

1. Now we apply rule (1) to Table 2. Here the only negative zj - cj is z_1-c_1 = -140/7 Hence x_1 should become a basic variable at the next iteration
2. We compute the minimum of the ratio

$$\text{Min} \left\{ \frac{184}{\frac{10}{7}}, \frac{45}{\frac{5}{7}}, \frac{116}{\frac{4}{7}} \right\} = \text{Min} \left\{ \frac{644}{5}, 63, 203 \right\} = 63$$

This minimum occurs corresponding to s_4, it becomes a non basic variable in next iteration.

3. Like Table 2, the Table 3 is computed using the rules (i), (ii), (iii) as described above.

Table-3

C_B	Basic Variables	C_j / X_B	60 / x_1	70 / x_2	0 / s_3	0 / s_4	0 / s_5
0	s3	94	0	0	1	-2	1
60	x1	63	1	0	0	7/5	-4/5
70	x2	80	0	1	0	-4/5	3/5
	z_j-c_j		0	0	0	28	-6

1. $z_5 – c_5 < 0$ should be made a basic variable in the next iteration.

2. Now compute the minimum ratios

$$\text{Min} \quad \frac{94}{1}, \ \frac{80}{\frac{3}{5}} \qquad = 94$$

Note: Since y_{25} = -4/5 < 0, the corresponding ratio is not taken for comparison. The variable s_3 becomes non basic in the next iteration.

3. From the Table 3, Table 4 is calculated following the usual steps.

Table-4

C_B	Basic Variables	C_j / X_B	60 / x_1	70 / x_2	0 / s_3	0 / s_4	0 / s_5
0	s_5	94	0	0	1	-2	1
60	x_1	691/5	1	0	4/5	-1/5	0
70	x_2	118/5	0	1	-3/5	2/5	0
	Z_j - C_j		0	0	6	16	0

Note that $z_j – c_j \geq 0$ for all j, so that the objective function can't be improved any further.

Thus, the objective function is maximized for x_1 = 691/5 and x_2=118/5 and the maximum value of the objective function is 9944.

1.4 Simplex Method with More Than Two Variables

An organization has three machine shops viz. A, B and C and it produces three product viz. X, Y and Z using these three machine shops. Each product involves the operation of the machine shops. The time available at the machine shops A, B and C are 100, 72 and 80 hours respectively. The profit per unit of product X, Y and Z is $22, $6 and $2 respectively. The following table shows the time required for each operation for unit amount of each product. Determine an appropriate product mix so as to maximize the profit.

	Machine			
Products	A	B	C	Profit per unit
X	10	7	2	22
Y	2	3	4	6
Z	1	2	1	2
Hours available	100	72	80	

Solution

First we have to develop linear programming formulation. The linear programming formulation of the product mix problem is:

Maximize

$22x_1 + 6x_2 + 2x_3$

Subject to:

$10\ x_1 + 2\ x_2 + x_3 \leq 100$

$7\ x_1 + 3\ x_2 + 2\ x_3 \leq 72$

$2\ x_1 + 4\ x_2 + x_3 \leq 80$

$x_1, x_2, x_3 \geq 0$

We introduce slack variables s4, s5 and s6 to make the inequalities equation.

Thus, the problem can be stated as

Maximize

$22x_1 + 6x_2 + 2\ x_3$

Subject to:

$10\ x_1 + 2\ x_2 + x_3 + s_4\ = 100$

$7\ x_1 + 3\ x_2 + 2\ x_3 + s_5\ = 72$

$2\ x_1 + 4\ x_2 + x_3 + s_6 = 80$

x_1, x_2, x_3, s4, s5, $s_6 \geq 0$

From the above equation the simplex Table 5 can be obtained in a straight forward manner. Here the basic variables are s_4, s_5 and s_6. Therefore CB1 = CB2 = CB3 = 0.

Table 5

CB	Basic Variable	C_j XB	22 x_1	6 x_2	2 x_3	0 s_4	0 s_5	0 s_6
0	s_4	100	10	2	1	1	0	0
0	s_5	72	7	3	2	0	1	0
0	s_6	80	2	4	1	0	0	1
	z_j-c_j		-22	-6	-2	0	0	0

1. Z_1- C_1 = -22 is the smallest negative value. Hence x1 should be taken as a basic variable in the next iteration.
2. Calculate the minimum of the ratios

$$\text{Min} \quad \frac{100}{10}, \frac{72}{7}, \frac{80}{2} \quad = 10$$

3. The variable s4 corresponding to which minimum occurs is made a non basic variable.

4. From the Table 5, the Table 6 is calculated using the following rules:

a) The revised basic variables are x_1, s_5, s_6. Accordingly we make CB1=22, CB2=0 and CB3=0.
b) Since x_1 is the incoming variable we make x_1 coefficient one by dividing each element of row 1 by 10. Thus the numerical value of the element corresponding to x_2 is 2/10, corresponding to x_3 is 1/10, corresponding to s_4 is 1/10, corresponding to s_5 is 0/10 and corresponding to s_6 is 0/10 in Table 6.
c) The incoming basic variable should only appear in the first row. So we multiply first row of Table 2 by 7 and subtract if from the second row of Table 1 element by element.

Thus,

The element corresponding to x_1 in the second row of Table 6 is zero

The element corresponding to x_2 is 3 – 7 * 2/10 = 16/10

By using this way we get the elements of the second and the third row in Table 6.

Similarly, the calculation of numerical values of basic variables in Table 6 is done.

Table-6

CB	Basic Variable	C_j XB	22 x_1	6 x_2	2 x_3	0 s_4	0 s_5	0 s_6
22	x_1	10	1	2/10	1/10	1/10	0	0
0	s_5	2	0	16/10	13/10	-7/10	1	0
0	s_6	60	0	18/5	4/5	-1/5	0	1
	z_j - c_j		0	-8/5	1/5	12/5	0	0

1. $z_2 - c_2 = -8/5$. So x_2 becomes a basic variable in the next iteration.
2. Calculate the minimum of the ratios

$$\text{Min} \quad \frac{10}{\frac{2}{10}}, \quad \frac{7}{\frac{16}{10}}, \quad \frac{60}{\frac{18}{5}} \quad = \text{Min} \quad 50, \frac{70}{16}, \frac{300}{18} \quad = 70/16$$

Hence the variable s5 will be a non basic variable in the next iteration.

3. From Table 6, the Table 7 is calculated using the rules (a), (b) and (c) mentioned above.

Table-7

CB	Basic Variable	C_j XB	22 x_1	6 x_2	2 x_3	0 s_4	0 s_5	0 s_6
22	x_1	73/8	1	0	-1/16	3/16	-1/8	0
6	x_2	30/8	0	1	13/16	-7/16	5/8	0
0	s_6	177/4	0	0	-17/8	11/8	-9/4	1
	z_j-c_j		0	0	24/16	24/16	1	0

Note that all $z_j – c_j \geq 0$, so that the solution is $x_1 = 73/8$, $x_2 = 30/8$ and $s_6 = 177/4$ maximizes the objective function.

The Maximum Profit is: 22*73/8 + 6*30/8 = 1606/8 + 180/8 = 1786/8 = 223.25

1.5 Two Phase and M-Method

In the last two section we discussed the simplex method was applied to linear programming problems with less than or equal to (≤) type constraints. Thus, there we could introduce slack variables which provide an initial basic feasible solution of the problem.

Generally, the linear programming problem can also be characterized by the presence of both 'less than or equal to' type or 'greater than or equal to (≥)' type constraints. In such case it is not always possible to obtain an initial basic feasible solution using slack variables.

The greater than or equal to type of linear programming problem can be solved by using the following methods:

1. Two Phase Method
2. M Method

In this section we will discuss these two methods.

1.6 Two Phase Method

We discuss the Two Phase Method with the help of the following Example:

Example:

Minimize

$12.5x_1 + 14.5x_2$

Subject to:

$x_1 + x_2 \geq 2000$

$0.4x_1 + 0.75x_2 \geq 1000$

$0.075x_1 + 0.1x_2 \leq 200$

$x_1, x_2 \geq 0$

Solution

Minimize

$12.5x_1 + 14.5x_2$

Subject to:

$x_1 + x_2 \geq 2000$

$0.4x_1 + 0.75x_2 \geq 1000$

$0.075x_1 + 0.1x_2 \leq 200$

$x_1, x_2 \geq 0$

Here the objective function is to be minimized; the values of x_1 and x_2 which minimized this objective function are also the values which maximize the revised objective function i.e.

Maximize

$-12.5x_1 - 14.5x_2$

We can multiply the second and the third constraints by 100 and 1000 respectively for the convenience of calculation.

Thus, the revised linear programming problem is:

Maximize

$-12.5x_1 - 14.5x_2$

Subject to:

$x_1 + x_2 \geq 2000$

$40x_1 + 75x_2 \geq 100000$

$75x_1 + 100x_2 \leq 200000$

$x_1, x_2 \geq 0$

Now we convert the two inequalities by introducing surplus variables s_3 and s_4 respectively. The third constraint is changed into an equation by introducing a slack variable s_5.

Thus, the linear programming problem becomes as

Maximize

$-12.5x_1 - 14.5x_2 = -25/2x_1 - 29/2x_2$

Subject to:

$x_1 + x_2 \ -s_3 = 2000$

$40x_1 + 75x_2 - s_4 = 100000$

$75x_1 + 100x_2 \ + s_5 = 200000$

$x_1, x_2, s_3, s_4, s_5 \geq 0$

Even though the surplus variables can convert greater than or equal to type constraints into equations they are unable to provide initial basic variables to start the simplex method calculation. So we may have to introduce two more additional variables a_6 and a_7 called as ***artificial variable*** to facilitate the calculation of an initial basic feasible solution.

In this method the calculation is carried out in two phases hence tow phase method.

Phase I

In this phase we will consider the following linear programming problem

Maximize

$-a_6 - a_7$

Subject to:

$x_1 + x_2 - s_3 \quad +a_6 \quad = 2000$

$40x_1 + 75x_2 \quad -s_4 \quad + a_7 = 100000$

$75x_1 + 100x_2 \quad +s_5 = 200000$

$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0$

The initial basic feasible solution of the problem is

$a_6 = 2000$, a_7=100000 and s5 = 200000.

As the minimum value of the objective function of the Phase I is zero at the end of the Phase I calculation both a_6 and a_7 become zero.

CB	Basic variables	C_j XB	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5	-1 a_6	-1 a_7
-1 -1 0	a_6 a_7 s_5	2000 100000 200000	1 40 75	1 75 100	-1 0 0	0 -1 0	0 0 1	1 0 0	0 1 0
		$z_j - c_j$	-41	-76	1	1	0	0	0

Here x_2 becomes a basic variable and a7 becomes non basic variable in the next iteration. It is no longer considered for re-entry in the table.

CB	Basic variables	C_j XB	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5	-1 a_6
-1	a_6	2000/3	7/15	0	-1	1/75	0	1
0	x_2	4000/3	8/15	1	0	-1/75	0	0
0	s_5	200000/3	65/3	0	0	4/3	1	0
		$z_j - c_j$	-1/15	0	1	-1/75	0	0

Then x_1 becomes a basic variable and a_6 becomes a non basic variable in the next iteration.

CB	Basic variables	C_j XB	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5
0	x_1	10000/7	1	0	-15/7	1/35	0
0	x_2	4000/7	0	1	8/7	-1/35	0
0	s_5	250000/7	0	0	325/7	16/21	1
		z_j-c_j	0	0	0	0	0

The calculation of Phase I end at this stage. Note that, both the artificial variable have been removed and also found a basic feasible solution of the problem.

The basic feasible solution is:

x_1 = 10000/7, x_2 = 4000/2, s_5 = 250000/7.

Phase II

The initial basic feasible solution obtained at the end of the Phase I calculation is used as the initial basic feasible solution of the problem. In this Phase II calculation the original objective function is introduced and the usual simplex procedure is applied to solve the linear programming problem.

CB	Basic variables	C_j XB	-25/2 x_1	-29/2 x_2	0 s_3	0 s_4	0 s_5
-25/2	x_1	10000/7	1	0	-15/7	1/35	0
-29/2	x_2	4000/7	0	1	8/7	-1/35	0
0	s_5	250000/7	0	0	325/7	5/7	1
		z_j - c_j	0	0	143/14	2/35	0

In this Table 1 all Zj - Cj ≥ 0 the current solution maximizes the revised objective function.

Thus, the solution of the problem is:

x_1 = 10000/7 = 1428 and x = 4000/7 = 571.4 and

The Minimum Value of the objective function is: 26135.3

1.7 M Method

In this method also we need artificial variables for determining the initial basic feasible solution. The M method is explained in the following Example with the help of the previous Example.

Example:

Maximize

$-12.5x_1 - 14.5x_2$

Subject to:

$x_1 + x_2 - s3 = 2000$

$40 x_1 + 75 x_2 - s_4 = 100000$

$75 x_1 + 100 x_2 + s_5 = 200000$

$x_1, x_2, s_3, s_4, s_5 \geq 0.$

Introduce the artificial variables a_6 and a_7 in order to provide basic feasible solution in the second and third constraints. The objective function is revised using a large positive number say M.

Thus, instead of the original problem, consider the following problem i.e.

Maximize:

$-12.5x_1 - 14.5x_2 - M (a_6 + a_7)$

Subject to:

$x_1 + x_2 - s_3 + a_6 = 2000$

$40x_1 + 75x_2 - s_4 + a_7 = 100000$

$75x_1 + 100x_2 + s_5 = 200000$

$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0.$

The coefficient of a_6 and a_7 are large negative number in the objective function. Since the objective function is to be maximized in the optimum solution, the artificial variables will be zero. Therefore, the basic variable of the optimum solution are variable other than the artificial variables and hence is a basic feasible solution of the original problem.

The successive calculation of simplex tables is as follows:

CB	Basic variables	C_j XB	-12.5 x_1	-14.5 x_2	0 s_3	0 s_4	0 s_5	-M a_6	-M a_7
-M	a_6	2000	1	1	-1	0	0	1	0

-M	a_7	100000	40	75	0	-1	0	0	1
0	s_5	200000	75	100	0	0	1	0	0
		z_j-c_j	-41M +12.5	-76M +14.5	M	M	0	0	0

Since M is a large positive number, the coefficient of M in the zj – cj row would decide the entering basic variable. As -76M < -41M, x2 becomes a basic variable in the next iteration replacing a_7. The artificial variable a_7 can't be re-entering as basic variable.

CB	Basic variables	C_j XB	-12.5 x_1	-14.5 x_2	0 s_3	0 s_4	0 s_5	-M a_6
-M	a_6	2000/3	7/15	0	-1	1/75	0	1
-14.5	x_2	4000/3	8/15	1	0	-1/75	0	0
0	s_5	200000/3	65/3	0	0	4/3	1	0
		z_j-c_j	-7/15M +143/30	0	M	-M/75 +29/150	0	0

Now x_1 becomes a basic variable replacing a_6. Like a_7 the variable a_6 also artificial variable so it can't be re-entering in the table.

CB	Basic variables	C_j XB	-12.5 x_1	-14.5 x_2	0 s_3	0 s_4	0 s_5
-12.5	x_1	10000/7	1	0	-15/7	1/35	0
-14.5	x_2	4000/7	0	1	8/7	-1/35	0
0	s_5	250000/7	0	0	325/7	16/21	1
		z_j-c_j	0	0	143/14	2/35	0

Hence

The optimum solution of the problem is x_1 = 10000/7, x_2 = 4000/7 and The Minimum Value of the Objective Function is: 26135.3

1.8 Multiple Solutions

The simplex method also helps in identifying multiple solutions of a linear programming problem. This is explained with the help of the following Example:

Example:

Consider the following linear programming problem.

Maximize

$2000x_1 + 3000x_2$

Subject to:

$6x_1 + 9x_2 \leq 100$

$2x_1 + x_2 \leq 20$

$x_1, x_2 \geq 0.$

Solution

Introduce the slack variables s_3 and s_4, so that the inequalities can be converted in to equation as follows:

$6x_1 + 9x_2 + s_3 = 100$

$2x_1 + x_2 + s_4 = 20$

$x_1, x_2, s_3, s_4 \geq 0.$

The computation of simple procedure and tables are as follows:

CB	Basic variables	C_j XB	2000 x_1	3000 x_2	0 s_3	0 s_4
0	s_3	100	6	9	1	0
0	s_4	20	2	1	0	1
		zj-cj	-2000	-3000	0	0

CB	Basic variables	Cj XB	2000 x1	3000 x2	0 s3	0 s4

0	x2	100/9	2/3	1	1/9	0
0	s4	80/9	4/3	0	-1/9	1
		zj-cj	0	0	3000/9	0

Here zj - cj ≥ 0 for all the variables so that we can't improve the simplex table any more. Hence it is optimum.

The optimum solution is $x_1 = 0$, $x_2 = 100/9$ and

The maximum value of the objective function is: 100000/3 = 33333.33.

However, the zj - cj value corresponding to the non basic variable x_1 is also zero. This indicates that there is more than one optimum solution for the problem exists.

In order to calculate the value of the alternate optimum solution we have to introduce x_1 as a basic variable replacing s_4. The next table shows the computation of this.

CB	Basic variables	C_j XB	2000 x_1	3000 x_2	0 s_3	0 s_4
3000	x_2	20/3	0	1	1/6	1/2
2000	x_1	20/3	1	0	-1/12	3/4
		z_j-c_j	0	0	1000/3	3000

Thus,

$x_1 = 20/3$, $x_2 = 20/3$ also maximize the objective function and

The Maximum value of the objective function is: 100000/3 = 33333.33

Thus, the problem has multiple solutions.

1.9 Unbounded Solution

In this section we will discuss how the simplex method is used to identify the unbounded solution. This is explained with the help of the following Example.

Example:

Consider the following linear programming problem.

Maximize

$5x_1 + 4x_2$

Subject to:

$x_1 – x_2 \leq 8$

$x_1 \leq 7$

$x_1, x_2 \geq 0.$

Solution:

Introduce the slack variables s3 and s4, so that the inequalities becomes as equation as follows:

$x_1 + s_3 = 7$

$x_1 – x_2 + s_4 = 8$

$x_1, x_2, s_3, s_4 \geq 0.$

The calculation of simplex procedures and tables are as follows:

CB	Basic variables	C_j XB	5 x_1	4 x_2	0 s_3	0 s_4
0	s_3	7	1	0	1	0
0	s_4	8	1	-1	0	1
		z_j-c_j	-5	-4	0	0

CB	Basic variables	C_j XB	5 x_1	4 x_2	0 s_3	0 s_4
5	x_1	7	1	0	1	0
0	s_4	1	0	-1	-1	1
		z_j-c_j	0	-4	5	0

Note that $z_2 - c_2 < 0$ which indicates x2 should be introduced as a basic variable in the next iteration. However, both $y12 \leq 0$, $y22 \leq 0$

Thus, it is not possible to proceed with the simplex method of calculation any further as we cannot decide which variable will be non basic at the next iteration. This is the criterion for unbounded solution.

NOTE: If in the course of simplex computation $zj - cj < 0$ but $yij \leq 0$ for all i then the problem has no finite solution.

But in this case we may observe that the variable x2 is unconstrained and can be increased arbitrarily. This is why the solution is unbounded.

1.10 Infeasible Solution

This section illustrates how to identify the infeasible solution using simplex method. This is explained with the help of the following Example.

Example:

Consider the following problem.

Minimize

$200x_1 + 300x_2$

Subject to:

$2x_1 + 3x_2 \geq 1200$

$x_1 + x_2 \leq 400$

$2x_1 + 3/2x_2 \geq 900$

$x_1, x_2 \geq 0$

Solution

Since it is a minimization problem we have to convert it into maximization problem and introduce the slack, surplus and artificial variables. The problem appears in the following manner after doing all these procedure.

Maximize

$-200x_1 - 300x_2$

Subject to:

$2x_1 + 3x_2 - s_3 + a_6 = 1200$

$x_1 + x_2 + s_4 = 400$

$2x_1 + 3/2x_2 - s_5 + a_7 = 900$

$x_1, x_2, s_3, s_4, s_5, a_6, a_7 \geq 0$

Here the a_6 and a_7 are artificial variables. We use two phase method to solve this problem.

Phase I

Maximize

$-a_6 - a_7$ Subject to:

$2x_1 + 3x_2 - s_3 + a_6 = 1200$

$x_1 + x_2 + s_4 = 400$

$2x_1 + 3/2x_2 - s_5 + a = 900$

$x_1, x_2, s_3, s_4, s_5, a, a_7 \geq 0$

The calculation of simplex procedures and tables are as follows:

CB	Basic variables	C_j XB	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5	-1 a_6	-1 a_7
-1	a_6	1200	2	3	-1	0	0	1	0
0	s_4	400	1	1	0	1	0	0	1
-1	a_7	900	2	3/2	0	0	-1	0	0
		z_j-c_j	-4	-9/2	1	0	1	0	0

CB	Basic variables	C_j XB	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5	-1 a_7
0	x_2	400	2/3	1	-1/3	0	0	0
0	s_4	0	1/3	0	1/3	1	0	0
-1	a_7	300	1	0	1/2	0	-1	1
		z_j-c_j	-1	0	-1/2	0	1	0

CB	Basic variables	C_j XB	0 x_1	0 x_2	0 s_3	0 s_4	0 s_5	-1 a_7
0	x_2	400	0	1	-1	-2	0	0
0	x_1	0	1	0	1	3	0	0
-1	a_7	300	0	0	-1/2	-3	-1	1
		z_j-c_j	0	0	1/2	3	1	0

Note that zj - cj ≥ 0 for all the variables but the artificial variable a_7 is still a basic variable. This situation indicates that the problem has no feasible solution.

1.11 Check Your Progress:

There are some activities to check your progress. Answer the followings:

1. Value of replacement ratio to be considered as outgoing variable must always be _____________
2. In an linear programming problem with maximization objective______________value in net evaluation row is considered as incoming variable
3. Simplex method is based on _______________
4. A solution in which a basic variable has solution value equal to zero is ______________
5. Whenever there is a tie in the replacement ratios (b_i/a_{ij}) to be selected, the next solution will be a __________________
6. Artificial variable are added to compliance the condition of ____________________

1.12 Summary

The Simplex method is very useful and appropriate method for solving linear programming problem having more than tow variables. The slack variables are introduced for less than or equal to type, surplus variables are introduce for greater than or equal to type of linear programming problem. The basic feasible solution is important in order to solve the problem using the Simplex method.

A basic feasible solution of a system with m-equations and n-variables has m non-negative variables called as basic variables and n-m variables with value zero known as non-basic variables. The objective function is maximized or minimized at one of the basic feasible solutions.

Surplus variables can’t provide the basic feasible solution instead artificial variables are used to get the basic feasible solutions and it initiate the simplex procedure. Two phase and M-Method are available to solve linear programming problem in these case.

The Simplex method also used to identify the multiple, unbounded and infeasible solutions.

1.13 Keywords

***Basic Variable*: ** Variable of a basic feasible solution has n non-negative value.
Non Basic Variable: Variable of a feasible solution has a value equal to zero.
Artificial Variable: A non-negative variable introduced to provide basic feasible solution and initiate the simplex procedures.
Slack Variable: A variable corresponding to a ≤ type constraint is a non-negative variable introduced to convert the inequalities into equations.
Surplus Variable: A variable corresponding to a ≥ type constraint is a non-negative variable introduced to convert the constraint into equations.
***Basic Solution*: ** System of m-equation and n-variables i.e. m<n is a solution where at least n-m variables are zero.
***Basic Feasible Solution*: ** System of m-equation and n-variables i.e. m<n is a solution where m variables are non-negative and n-m variables are zero.
Optimum Solution: A solution where the objective function is minimized or maximized.

1.14 Self Assessment Test

Q1. A soft drinks company has a two products viz. Coco-cola and Pepsi with profit of $2 an $1 per unit. The following table illustrates the labour, equipment and materials to produce per unit of each product. Determine suitable product mix which maximizes the profit using simplex method.

Products	Pepsi	Cola	Total resources
Labour	3	2	12
Equipments	1	2.3	6.9
Resource	1	1.4	4.9

Q2. A factory produces three using three types of ingredients viz. A, B and C in different proportions. The following table shows the requirements o various ingredients as inputs per kg of the products.

	Ingredients		
Products	A	B	C

1	4	8	8
2	4	6	4
3	8	4	0

The three profits coefficients are 20, 20 and 30 respectively. The factory has 800 kg of ingredients A, 1800 kg of ingredients B and 500 kg of ingredient C.
Determine the product mix which will maximize the profit and also find out maximum profit.

Q3. Solve the following linear programming problem using two phase and M method.

Maximize
$12x_1 + 15x_2 + 9x_3$

Subject to:
$8x_1 + 16x_2 + 12x_3 \leq 250$
$4x_1 + 8x_2 + 10x_3 \geq 80$
$7x_1 + 9x_2 + 8x_3 = 105$
$x_1, x_2, x_3 \geq 0$

Q4. Solve the following linear programming problem using simplex method.
Maximize
$3x_1 + 2x_2$
Subject to:
$x_1 - x_2 \leq 1$
$x_1 + x_2 \geq 3$
$x_1, x_2 \geq 0$

Q5. Solve the following linear programming problem using simplex method.
Maximize
$x_1 + x_2$

Subject to:
$-2x_1 + x_2 \leq 1$
$x_1 \quad \leq 2$
$x_1 + x_2 \leq 3$
$x_1, x_2, x_3 \geq 0$

Q6. Maximize

$P = 3x_1 + 4x_2 + x_3$

Subject to:

$x_1 + 2x_2 + x_3 \leq 6$

$2x_1+2x_3 \leq 4$

$3x_1 + x_2 + x_3 \leq 9$

$x_1, x_2, x_3 \geq 0$

1.15 Answers to Check Your Progress

1. Non-negative
2. Highest non-negative
3. Iteration
4. Degenerate solution
5. Degenerate solution
6. Identity matrix

1.16 References/ Suggested Readings

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.:4	Vetter:
Linear Programming Problem (LPP): Dual Linear Programming Problems	

Structure:

1.1 Introduction

1.2 Dual Problem Formulation

1.3 Dual Problem Properties

1.4 Simple Way of Solving Dual Problem

1.5 Important characteristics of Duality

1.6 Advantages and Applications of Duality

1.7 Check your progress

1.8 Summary

1.9 Keywords

1.10 Self Assessment Test

1.11 Answers to check your progress

1.12 References/ Suggested Readings

Learning Objectives:

After Studying this lesson, students will be able to:

1. Understand the Dual Linear programming Problem
2. Formulate a Dual Problem
3. Solve the Dual Linear Programming Problem
4. Understand the Properties of a Dual Problem

1.1 Introduction

For every linear programming problem there is a corresponding linear programming problem called the dual. If the original problem is a maximization problem then the dual problem is minimization problem and if the original problem is a minimization problem then the dual problem is maximization problem. In either case the final table of the dual problem will contain both the solution to the dual problem and the solution to the original problem.
The theory of duality is a very elegant and important concept within the field of operations research. This theory was first developed in relation to linear programming, but it has many applications, and perhaps even a more natural and intuitive interpretation, in several related areas such as nonlinear programming, networks and game theory.

The notion of duality within linear programming asserts that every linear program has associated with it a related linear program called its dual. The original problem in relation to its dual is termed the primal. It is the relationship between the primal and its dual, both on a mathematical and economic level, that is truly the essence of duality theory.

The solution of the dual problem is readily obtained from the original problem solution if the simplex method is used.

The formulation of the dual problem also sometimes referred as the concept of duality is helpful for the understanding of the linear programming. The variable of the dual problem is known as the dual variables or shadow price of the various resources. The dual problem is easier to solve than the original problem. The dual problem solution leads to the solution of the original problem and thus efficient computational techniques can be developed through the concept of duality. Finally, in the competitive strategy problem solution of both the original and dual problem is necessary to understand the complete problem.

Every LPP called the **primal** is associated with another LPP called **dual**. Either of the problems is primal with the other one as dual. The optimal solution of either problem reveals the information about the optimal solution of the other.

Let the primal problem be
Max $Z_x = c_1x_1 + c_2x_2 + \dots + c_nx_n$
Subject to restrictions

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$

.

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_n$

and $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

The corresponding dual is defined as

Min $Z_w = b_1w_1 + b_2w_2 + . b_mw_m$

Subject to restrictions

$a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \geq c_1$

$a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \geq c_2$

.

.

$a_{1n}w_1 + a_{2n}w_2 + \ldots\ldots\ldots + a_{mn}w_m \geq c_n$

and

$w_1, w_2, \ldots, w_m \geq 0$

1.2 Dual Problem Formulation

If the original problem is in the standard form then the dual problem can be formulated using the following rules:

1. The number of constraints in the original problem is equal to the number of dual variables;
2. The number of constraints in the dual problem is equal to the number of variables in the original problem;
3. The original problem profit coefficients appear on the right hand side of the dual problem constraints;
4. If the original problem is a maximization problem then the dual problem is a minimization problem. Similarly, if the original problem is a minimization problem then the dual problem is a maximization problem;
5. The original problem has less than or equal to (≤) type of constraints while the dual problem has greater than or equal to (≥) type constraints;
6. The coefficients of the constraints of the original problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

The Dual Linear Programming Problem is explained with the help of the following Example:

Example:

Consider the following product mix problem:

Three machine shops A, B, C produces three types of products X, Y, Z respectively. Each product involves operation of each of the machine shops. The time required for each operation on various products is given as follows:

		Machine		
Products	A	B	C	Profit per unit($)
X	10	7	2	12
Y	2	3	4	6
Z	1	2	1	1
Hours available	100	77	80	

The available hours at the machine shops A, B, C are 100, 77, and 80 only. The profit per unit of products X, Y, and Z is $12, $3, and $1 respectively.

Solution:

The formulation of Linear Programming (original problem) is as follows:

Maximize

$12x_1 + 3x_2 + x_3$

Subject to:

$10x_1 + 2x_2 + x_3 \leq 100$

$7x_1 + 3x_2 + 2x_3 \leq 77$

$2x_1 + 4x_2 + x_3 \leq 80$

$x_1, x_2, x_3 \geq 0$

We introduce the slack variables s_4, s_5 and s_6 then the equalities becomes as:

Maximize

$12x_1 + 3x_2 + x_3$

Subject to:

$10x_1 + 2x_2 + x_3 + s_4 = 100$

$7x_1 + 3x_2 + 2x_3 + s_5 = 77$

$2x_1 + 4x_2 + x_3 + s_6 = 80$

$x_1, x_2, x_3, s_4, s_5, s_6 \geq 0$

Form the above equations, the first simplex table-1 is obtained is as follows:

Table-1

CB	Basic Variable	C_j XB	12 x_1	3 x_2	1 x_3	0 s_4	0 s_5	0 s_6
0	s_4	100	10	2	1	1	0	0
0	s_5	77	7	3	2	0	1	0
0	s_6	80	2	4	1	0	0	1
	Zj - Cj		-12	-3	-1	0	0	0

1. The smallest negative element in the above table of z1 – c1 is -12. Hence, x1 becomes a basic variable in the next iteration.
2. Determine the minimum ratios

$$\text{Min} \quad \frac{100}{10}, \frac{72}{7}, \frac{80}{2} \quad = 10$$

Here the minimum value is s_4, which is made as a non-basic variable.

3. The next Table 2 is calculated using the following rules:

(i) The revised basic variables are x1, s_5, s_6. Accordingly we make CB1=22, CB2=0 and CB3=0.

(ii) Since x1 is the incoming variable we make x1 coefficient one by dividing each element of row 1 by 10. Thus the numerical value of the element corresponding to x_2 is 2/10, corresponding to x_3 is 1/10, corresponding to s_4 is 1/10, corresponding to s_5 is 0/10 and corresponding to s_6 is 0/10 in Table 2.

(iii) The incoming basic variable should only appear in the first row. So we multiply first row of Table 2 by 7 and subtract if from the second row of Table 1 element by element.

Thus,

The element corresponding to x1 in the second row of Table 2 is zero

The element corresponding to x_2 is $3 – 7 * 2/10 \ = 16/10$

By using this way we get the elements of the second and the third row in Table 2. Similarly, the calculation of numerical values of basic variables in Table 2 is done.

Table- 2

CB	Basic Variable	C_j XB	22 x_1	6 x_2	2 x_3	0 s_4	0 s_5	0 s_6
12	x_1	10	1	2/10	1/10	1/10	0	0
0	s_5	7	0	16/10	13/10	-7/10	1	0
0	s_6	60	0	18/5	4/5	-1/5	0	1
	z_j-c_j		0	-3/5	1/5	6/5	0	0

4. $z_2 - c_2 = -3/5$. So x_2 becomes a basic variable in the next iteration.
5. Determine the minimum of the ratios

$$\text{Min} = \frac{10}{\frac{2}{10}}, \frac{7}{\frac{16}{10}}, \frac{60}{\frac{18}{5}} \qquad \text{Min} = 50, \frac{70}{16}, \frac{300}{18} = 70/16$$

So that, the variable s_5 will be a non- basic variable in the next iteration

6. From Table 2, the Table 3 is calculated using the rules (i), (ii) and (iii) mentioned above.

Table-3

CB	Basic Variable	C_j XB	12 x_1	3 x_2	1 x_3	0 s_4	0 s_5	0 s_6
12	x_1	73/8	1	0	-1/16	3/16	-1/8	0
3	s_5	35/8	0	1	13/16	-7/16	5/8	0
0	s_6	177/4	0	0	-17/8	11/8	-9/4	1
	z_j-c_j		0	0	11/16	15/16	3/8	0

Since all the $z_i - c_j \geq 0$, the optimum solution is as:

$x_1 = 73/8$ and $x_2 = 35/8$ and The Maximum Profit is: \$981/8 = \$122.625

Suppose an investor is deciding to purchase the resources A, B, C. What offers is he going to produce?

Let, assume that W1, W2 and W3 are the offers made per hour of machine time A, B and C respectively. Then these prices W1, W2 and W3 must satisfy the conditions given below:

A. $W_1, W_2, W_3 \geq 0$

B. Assume that the investor is behaving in a rational manner; he would try to bargain as much as possible so that the total annual payable to the produces would be as little as possible. This leads to the following condition:

Minimize

$100W_1 + 77W_2 + 80W_3$

C. The total amount offer by the investor to the three resources viz. A, B and C required to produce one unit of each product must be at least as high as the profit gained by the producer per unit.

Since, these resources enable the producer to earn the specified profit corresponding to the product he would not like to sell it for anything less assuming he is behaving rationally. This leads to the following conditions:

$10w_1 + 7w_2 + 2w_3 \geq 12$

$2 w_1 + 3 w_2 + 4 w_3 \geq 3$

$w_1 + 2 w_2 + w_3 \geq 1$

Thus, in this case we have a linear problem to ascertain the values of the variable w_1, w_2, w_3. The variables w_1, w_2 and w_3 are called as ***dual variables***.

Note:

The original (primal) problem illustrated in this example

a. considers the objective function maximization

b. contains $\leq$ type constraints

c. has non-negative constraints

This original problem is called as primal problem in the standard form.

1.3 Dual Problem Properties

The following are the different properties of dual programming problem:

(i.) if the original problem is in the standard form, then the dual problem solution is obtained from the zj – cj values of slack variables.

For example: In the above example, the variables s4, s4 and s6 are the slack variables. Hence the dual problem solution is w1 = $z_4 - c_4$ = 15/16, w2 = $z_5 - c_5$ = 3/8 and w3 = $z_6 - c_6$ = 0.

(ii) The original problem objective function maximum value is the minimum value of the dual problem objective function.

For example:

From the above example, we know that the original problem maximum values is 981/8 = 122.625. So that the minimum value of the dual problem objective function is

100*15/16 + 77*3/8 + 80*0 = 981/8

Here the result has an important practical implication. If both producer and investor analyzed the problem then neither of the two can outmaneuver the other.

(iii) Shadow Price: A resource shadow price is its unit cost, which is equal to the increase in profit to be realized by one additional unit of the resource.

For example:

Let the minimum objective function value is expressed as:

100 * 15/16 + 77 * 3/8 + 80 * 0

If the first resource is increased by one unit the maximum profit also increases by 15/16, which is the first dual variable of the optimum solution. Therefore, the dual variables are also referred as the resource shadow price or imputed price. Note that in the previous example the shadow price of the third resource is zero because there is already an unutilized amount, so that profit is not increased by more of it until the current supply is totally exhausted.

(iv) In the originals problem, if the number of constraints and variables is m and n then the constraint and variables in the dual problem is n and m respectively. Suppose the slack variables in the original problem is represented by y1, y1, ….., yn and the surplus variables are represented by $z_1, z_2, \ldots, z_n$ in the dual problem.

(v) Suppose, the original problem is not in a standard form, then the dual problem structure is unchanged. However, if a constraint is greater than or equal to type, the corresponding dual variable is negative or zero. Similarly, if a constraint in the original problem is equal to type, then the corresponding dual variables is unrestricted in sign.

1.4 Simple Way of Solving Dual Problem

Example:

Consider the following linear programming problem

Maximize
$22x_1 + 25x_2 + 19x_3$

Subject to:
$18 x_1 + 26 x_2 + 22 x_3 \leq 350$

$14 x_1 + 18 x_2 + 20 x_3 \geq 180$

$17 x_1 + 19 x_2 + 18 x_3 = 205$

$x_1, x_2, x_3 \geq 0$

Note that this is a primal or original problem.

The corresponding dual problem for this problem is as follows:

Minimize
$250w_1 + 80w_2 + 105w_3$

Subject to:
$18 w_1 + 4 w_2 + 7 w_3 \geq 22$

$26 w_1 + 18 w_2 + 19 w_3 \geq 25$

$22 w_1 + 20 w_2 + 18 w_3 \geq 19$

$w_1 \geq 0$, $w_2 \leq$, and w_3 is unrestricted in sign (+ or -).

Now, we can solve this using simplex method as usual.

Example:

Minimize
$P = x_1 + 2x_2$

Subject to:
$x_1 + x_2 \geq 8$

$2x_1 + x_2 \geq 12$

$x_1 \geq 1$

Solution:

Step 1: Set up the P-matrix and its transpose

$$P = \begin{pmatrix} 1 & 1 & 8 \\ 2 & 1 & 12 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad PT = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 8 & 12 & 1 & 0 \end{pmatrix}$$

Step 2: Develop the constraints and objective function for the dual

$w_1 + 2w_2 + w_3 \leq 1$

$w_1 + w_2 \leq 2$

$z = 8w_1 = 12w_2 + 2$

Step 3: Construct the initial simplex tableau for the dual

w_1	w_2	w_3	s_1	s_2	g	z
1	2	1	1	0	0	1
1	1	0	0	1	0	2
-8	-12	-1	0	0	1	0

Since there are no negative entries in the last column above the third row, we have a standard simplex problem. The most negative number in the bottom row to the left of the last column is -12. This establishes the pivot column. The smallest non-negative ratio is 1/2. The pivot element is 2 in the w_2-column.

Step 4: Pivoting

Pivoting about 2, we get

w_1	w_2	w_3	s_1	s_2	g	z
1	2	1	1	0	0	1

½	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

w_1	w_2	w_3	s_1	s_2	g	Z
½	**1**	1/2	1/2	0	0	½
1	1	0	0	1	0	2
-8	-12	-1	0	0	1	0

w_1	w_2	w_3	s_1	s_2	g	Z
½	**1**	1/2	1/2	0	0	½
½	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

The most negative entry in the bottom row to the left of the last column is -2. The smallest non- negative ratio is the 1/2 in the first row. This is the next pivot element.

Pivoting about 1/2

w_1	w_2	w_3	s_1	s_2	g	Z
½	1	1/2	1/2	0	0	½
½	0	-1/2	-1/2	1	0	3/2
-2	0	5	6	0	1	6

Since there are no negative entries in the bottom row and to the left of the last column, the process is complete. The solutions are at the feet of the slack variable columns.

Therefore, the optimum solution provided by x_1= 8 and x_2 = 0 and the Minimum Value is: 8

w_1	w_2	w_3	s_1	s_2	G	Z
1	2	1	1	0	0	1
0	-1	-1	-1	1	0	1

0	4	7	8	0	1	8

1.5 Important characteristics of Duality

1) Dual of dual is primal
2) If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.
3) If any of the two problems has an infeasible solution, then the value of the objective function of the other is unbounded.
4) The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual
5) If either the primal or dual has an unbounded solution, then the solution to the other problem is infeasible.
6) If the primal has a feasible solution, but the dual does not have then the primal will not have a finite optimum solution and vice versa.

1.6 Advantages and Applications of Duality

1. Sometimes dual problem solution may be easier than primal solution, particularly when the number of decision variables is considerably less than slack / surplus variables.
2. In the areas like economics, it is highly helpful in obtaining future decision in the activities being programmed.
3. In physics, it is used in parallel circuit and series circuit theory.
4. In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains.
5. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
6. When a problem does not yield any solution in primal, it can be verified with dual.
7. Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

1.7 Check Your Progress:

1. The number of constraints in the original problem is equal to the number of ________________
2. The number of constraints in the dual problem is equal to the number of variables in the

__________;

3. The original problem profit coefficients appear on the _________ of the dual problem constraints;
4. If the original problem is a maximization problem then the dual problem is a ______________________
5. The original problem has less than or equal to (≤) type of constraints while the dual problem has _____________________ type constraints;

1.8 Summary

Duality in linear programming is essentially a unifying theory that develops the relationships between a given linear program and another related linear program stated in terms of variables with this shadow-price interpretation.

For every linear programming problem there is a dual problem. The variables of the dual problem are called as dual variables. The variables have economic value, which can be used for planning its resources. The dual problem solution is achieved by the simplex method calculation of the original (primal) problem. The dual problem solution has certain properties, which may be very useful for calculation purposes.

Steps for a Standard Primal Form

Step 1 – Change the objective function to Maximization form

Step 2 – If the constraints have an inequality sign '≥' then multiply both sides by -1 and convert the inequality sign to '≤'.

Step 3 – If the constraint has an '=' sign then replace it by two constraints involving the inequalities going in opposite directions. For example $x_1 + 2x_2 = 4$ is written as

$x_1+2x_2 \leq 4$

$x_1+2x_2 \geq 4$ (using step2) → $- x_1-2x_2 \leq - 4$

Step 4 – Every unrestricted variable is replaced by the difference of two non-negative variables.

Step5 – We get the standard primal form of the given LPP in which.

All constraints have '≤' sign, where the objective function is of maximization form.

All constraints have '≥' sign, where the objective function is of minimization from.

Rules for Converting any Primal into its Dual

1. Transpose the rows and columns of the constraint co-efficient.

2. Transpose the co-efficient ($c_1,c_2,\ldots c_n$) of the objective function and the right side constants ($b_1,b_2,\ldots b_n$)

3. Change the inequalities from '≤' to '≥' sign.

4. Minimize the objective function instead of maximizing it.

1.9 Keywords

Primal**:** This is the original linear programming problem, also called as primal problem.

Dual Problem**:** A dual problem is a linear programming problem is another linear programming problem formulated from the parameters of the primal problem.

Dual Variables**:** Dual programming problem variables.

Optimum Solution: The solution where the objective function is minimized or maximized.

Shadow Price**:** Price of a resource is the change in the optimum value of the objective function per unit increase of the resource.

1.10 Self Assessment Test

Q1. An organization manufactures three products viz. A, B and C. The required raw material per piece of product A, B and C is 2kg, 1kg, and 2kg. Assume that the total weekly availability is 50 kg. In order to produce the products the raw materials are processed on a machine by the labour force and on a weekly availability of machine hours is 30. Assume that the available total labour hour is 26. The following table illustrates time required per unit of the three products.

Product	Labour Hour	Machine Hour
A	0.5	1
B	3	2
C	1	1

The profit per unit from the products A, B and C are #25, #30 and #40.

Formulate the dual linear programming problem and determine the optimum values of the dual variables.

Q2. Write the dual to following problem

Minimize
$3w_1 + 4w_2$

Subject to:
$3\ w_1 + 4\ w_2 \geq 24$

$2\ w_1 + w_2 \geq 10$

$5\ w_1 + 3\ w_2 \geq 29$

$w_1,\ w_2 \geq 0$

Q 3 Write the dual to following problem

Minimize
$Z = 2x_2 + x_3$

Subject to
$x_1+x_2 \geq 2$

$2x_1+x_2+6x_3 \leq 6$

$x_1 - x_2 +3x_3 = 4$

$x_1,\ x_2\ ,\ x_3 \geq 0$

Q4. Write the dual to following problem

Minimize
$Z = 3x_1- 2x_2 + 4x_3$

Subject to
$3x_1+5x_2 + 4x_3 \geq 7$

$6x_1+x_2+3x_3 \geq 4$

$7x_1 - 2x_2 - x_3 \geq$

$10\, x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Q5. Write the dual to following problem

Maximize
$Z = 2x_1 + 3x_2 + x_3$

Subject to
$4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 5x_3 = 4$

$x_1, x_2 \geq 0$

1.11 Answers to check your progress

1. Dual variable
2. Original problem
3. Right hand side
4. Minimization problem
5. greater than or equal to (≥)

1.12 References/ Suggested Readings

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.:5	Vetter:
Linear Programming Problem (LPP): Sensitivity Analysis in Linear Programming Problems	

Structure:

1.1 Introduction

1.2 Intuition and Overview

1.3 Changing Objective Function

1.4 Changing a Right-Hand Side Constant

1.5 Adding a Constraint

1.6 Relationship to the Dual

1.7 Understanding Sensitivity Information Provided by Excel

1.8 Formulation and Solution

1.9 Sensitivity Analysis for Objective Function Coefficients Using Excel

1.10 Check your progress

1.11 Summary

1.12 Key Words

1.13 Self Assessment Test

1.14 Answers to check your progress

1.15 References/ Suggested Readings

Learning Objectives:

After Studying this lesson, students will be able to:

1. Understand the sensitivity analysis in Linear programming Problem
2. Understand the applications of sensitivity analysis
3. Solve the problem while the values of variables are changed
4. Understand the properties of sensitivity test

1.1 Introduction

After solving a lengthy LPP, one may notice that certain curtail variables either from objective function or from constraints may change over time. Such an observation definitely poses the questions whether the already obtained optimum solution is still valid or not. One way out to this question is simple. We redo the problem for optimality by incorporating the noticed changes into the original problem. The other option is to conduct sensitivity tests for the validity of the old results under new situation. Such a test will give us a range in which the variations in the concerned variables leave the already obtained optimum values of the decision variables unaltered. From the final simplex tableau, many such important post optimality questions can be answered.

When you use a mathematical model to describe reality you must make approximations. The world is more complicated than the kinds of optimization problems that we are able to solve. Linearity assumptions usually are significant approximations. Another important approximation comes because you cannot be sure of the data that you put into the model. Your knowledge of the relevant technology may be imprecise, forcing you to approximate values in A, b, or c (coefficients of objective functions and constant values on right hand side in constraint equations). Moreover, information may change.

Sensitivity analysis is a systematic study of how sensitive solutions are to small changes in the data. This is also known as post optimality analysis. The basic idea is to be able to give answer to questions of the form:

1. If the objective function changes, how does the solution change?
2. If resources available change, how does the solution change?
3. If a constraint is added to the problem, how does the solution change?

One approach to these questions is to solve lots of linear programming problems. For example, if you think that the price of your primary output will be between $100 and $120 per unit, you can solve twenty different problems (one for each whole number between $100 and $120). This method would work, but it is inelegant and (for large problems) would involve a large amount of computation time. (In fact, the computation time is cheap, and computing solutions to similar problems is a standard technique for studying sensitivity in practice.).

The approach that is discussed in these notes takes full advantage of the structure of LP programming problems and their solution. It turns out that you can often figure out what happens in "nearby" linear programming problems just by thinking and by examining the information provided by the simplex algorithm. In this section, the sensitivity analysis information provided in Excel computations will be described. It will help you to get an intuition for the results.

1.2 Intuition and Overview

Throughout these notes you should imagine that you must solve a linear programming problem, but then you want to see how the answer changes if the problem is changed. In every case, the results assume that only one thing about the problem changes. That is, in sensitivity analysis you evaluate what happens when only one parameter of the problem changes.
To fix ideas, you may think about a LPP as given below:
Maximize
$2X_1 + 4X_2 + 3X_3 + X_4$

Subject to
$3X_1 + X_2 + X_3 + 4X_4 \leq 12$
$X_1 - 3X_2 + 2X_3 + 3X_4 \leq 7$
$2X_1 + X_2 + 3X_3 - X_4 \leq 10$

On solving it, Solution to this problem is Max value = 42, $x_1 = 0$; $x_2 = 10.4$; $x_3 = 0$; $x_4 = 0.4$

1.3 Changing Objective Function

Suppose that you solve an LP and then wish to solve another problem with the same constraints but a slightly different objective function. (I will always make only one change in the problem at a time. So if I change the objective function, not only will I hold the constraints fixed, but I will change only one coefficient in the objective function.)

When you change the objective function it turns out that there are two cases to consider. The first case is the change in a non-basic variable (a variable that takes on the value zero in the solution). In the example, the relevant non-basic variables are x_1 and x_3.

What happens to your solution if the coefficient of a non-basic variable decreases? For example, suppose that the coefficient of x_1 in the objective function above was reduced from

2 to 1 (so that the objective function is: max $x_1 + 4x_2 + 3x_3 + x_4$). What has happened is this: You have taken a variable that you didn't want to use in the first place (you set $x_1 = 0$) and then made it less profitable (lowered its coefficient in the objective function). You are still not going to use it. The solution does not change. Observation, If you lower the objective function coefficient of a non-basic variable, then the solution does not change.

What if you raise the coefficient? Intuitively, raising it just a little bit should not matter, but raising the coefficient a lot might induce you to change the value of x in a way that makes $x_1 > 0$. So, for a non-basic variable, you should expect a solution to continue to be valid for a range of values for coefficients of non- basic variables. The range should include all lower values for the coefficient and some higher values. If the coefficient increases enough (and putting the variable into the basis is feasible) then the solution changes

What happens to your solution if the coefficient of a basic variable (like x_2 or x_4 in the example) decreases? This situation differs from the previous one in that you are using the basis variable in the first place. The change makes the variable contribute less to profit. You should expect that a sufficiently large reduction makes you want to change your solution (and lower the value the associated variable).

For example, if the coefficient of x_2 in the objective function in the example were 2 instead of 4 (so that the objective was max $2x_1 +2x_2 +3x_3 +x_4$), maybe you would want to set $x_2 = 0$ instead of $x_2 = 10.4$. On the other hand, a small reduction in x_2's objective function coefficient would typically not cause you to change your solution. In contrast to the case of the non-basic variable, such a change will change the value of your objective function. You compute the value by plugging in x into the objective function, if $x_2 = 10.4$ and the coefficient of x_2 goes down from 4 to 2, then the contribution of the x_2 term to the value goes down from 41.6 to 20.8 (assuming that the solution remains the same).

If the coefficient of a basic variable goes up, then your value goes up and you still want to use the variable, but if it goes up enough, you may want to adjust x so that it x_2 is even possible. In many cases, this is possible by finding another basis (and therefore another solution). So, intuitively, there should be a range of values of the coefficient of the objective function (a range that includes the original value) in which the solution of the problem does not change. Outside of this range, the solution will change (to lower the value of the basic variable for reductions and increase its value of increases in its objective function coefficient). The value of the problem always changes when you change the coefficient of a basic variable.

1.4 Changing a Right-Hand Side Constant

We discussed this topic when we talked about duality. It is argued that dual prices capture the effect of a change in the amounts of available resources. When you changed the amount of resource in a non-binding constraint, then increases never changed your solution. Small decreases also did not change anything, but if you decreased the amount of resource enough to make the constraint binding, your solution could change. (Note the similarity between this analysis and the case of changing the coefficient of a non-basic variable in the objective function. Changes in the right-hand side of binding constraints always change the solution (the value of x must adjust to the new constraints). We saw earlier that the dual variable associated with the constraint measures how much the objective function will be influenced by the change.

1.5 Adding a Constraint

If you add a constraint to a problem, two things can happen. Your original solution satisfies the constraint or it doesn't. If it does, then you are finished. If you had a solution before and the solution is still feasible for the new problem, then you must still have a solution. If the original solution does not satisfy the new constraint, then possibly the new problem is infeasible. If not, then there is another solution. The value must go down. (Adding a constraint makes the problem harder to satisfy, so you cannot possibly do better than before). If your original solution satisfies your new constraint, then you can do as well as before. If not, then you will do worse. There is a rare case in which originally your problem has multiple solutions, but only some of them satisfy the added constraint. In this case, which you need not worry about,

1.6 Relationship to the Dual

The objective function coefficients correspond to the right-hand side constants of resource constraints in the dual. The primal's right-hand side constants correspond to objective function coefficients in the dual. Hence the exercise of changing the objective function's coefficients is really the same as changing the resource constraints in the dual. It is extremely useful to become comfortable switching back and forth between primal and dual relationships.

1.7 Understanding Sensitivity Information Provided by Excel

The main purpose of sensitivity analysis is to see the effect of frequent changes in the various coefficients and constant values on optimum solution. Since, for every change, we cannot go for solving the entire Linear programming again and again, therefore, use of Excel becomes very important in sensitivity analysis to take a decision.

Excel permits you to create a sensitivity report with any solved LP. The report contains two tables, one associated with the variables and the other associated with the constraints. In reading these notes, keep the information in the sensitivity tables associated with the first simplex algorithm example nearby.

Sensitivity Information on Changing (or Adjustable) Cells

The top table in the sensitivity report refers to the variables in the problem. The first column (Cell) tells you the location of the variable in your spreadsheet; the second column tells you its name (if you named the variable); the third column tells you the final value; the fourth column is called the reduced cost; the fifth column tells you the coefficient in the problem; the final two columns are labeled "allowable increase" and "allowable decrease." Reduced cost, allowable increase, and allowable decrease are new terms. They need definitions.

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The allowable increases and decreases are easier. We will discuss them first. The allowable increase is the amount by which you can increase the coefficient of the objective function without causing the optimal basis to change. The allowable decrease is the amount by which you can decrease the coefficient of the objective function without causing the optimal basis to change.

Take the first row of the table for the example. This row describes the variable x_1. The coefficient of x_1 in the objective function is 2. The allowable increase is 9, the allowable decrease is "1.00E+30," which means 1030, which really means. This means that provided that the coefficient of x_1 in the objective function is less than 11 = 2 + 9 = original value + allowable increase, the basis does not change. Moreover, since x1 is a non-basic variable, when the basis stays the same, the value of the problem stays the same too. The information in this line confirms the intuition provided earlier and adds something new. What is confirmed is that if you lower the objective coefficient of a non-basic variable, then your solution does not change. (This means that the allowable decrease will always be infinite for a non-basic variable.). The example also demonstrates your value will stay the same.

Increasing the coefficient of a non-basic variable may lead to a change in basis. In the example, if you increase the coefficient of x_1 from 2 to anything greater than 9 (that is, if you add more than the allowable increase of 7 to the coefficient), then you change the solution. The sensitivity table does not tell you how the solution changes, but common sense suggests that x_1 will take on a positive value. Notice that the line associated with the other non-basic variable of the example, x_3, is remarkably similar. The objective function coefficient is different (3 rather than 2), but the allowable increase and decrease are the same as in the row for x_1. It is a coincidence that the allowable increases are the same. It is no coincidence that the allowable decrease is the same. We can conclude that the solution of the problem does not change as long as the coefficient of x_3 in the objective function is less than or equal to 10.

Consider now the basic variables. For x_2, the allowable increase is infinite while the allowable decrease is 2.69. This means that if the solution won't change if you increase the coefficient of x_2, but it will change if you decrease the coefficient enough (that is, by more than 2.7). The fact that your solution does not change no matter how much you increase x_2's coefficient means that there is no way to make $x_2 > 10.4$ and still satisfy the constraints of the problem. The fact that your solution does change when you increase x_2's coefficient by enough means that there is a feasible basis in which x_2 takes on a value lower than 10.4. (You knew that. Examine the original basis for the problem.) The range for x_4 is different.

Line four of the sensitivity table says that the solution of the problem does not change provided that the coefficient of x_4 in the objective function stays between 16 (allowable increase 15 plus objective function coefficient 1) and -4 (objective function coefficient minus allowable decrease). That is, if you make x_4 sufficiently more attractive, then your solution will change to permit you to use more x_4. If you make x_4 sufficiently less attractive the solution will also change. This time to use less x_4. Even when the solution of the problem does not change, when you change the coefficient of a basic variable the value of the problem will change. It will change in a predictable way. Specifically, you can use the table to tell you the solution of the LP when you take the original constraints and replace the original objective function by

Maximize: $2x_1 + 6x_2 + 3x_3 + x_4$

That is, you change the coefficient of x_2 from 4 to 6, the solution to the problem remains the same. The value of the solution changes because now you multiply the 10.4 units of x_2 by 6 instead of 4. The objective function therefore goes up by 20.8.

The reduced cost of a variable is the smallest change in the objective function coefficient needed to arrive at a solution in which the variable takes on a positive value when you solve the problem. This is a mouthful. Fortunately, reduced costs are redundant information. The reduced cost is the negative of the allowable increase for non-basic variables (that is, if you change the coefficient of x_1 by −7, then you arrive at a problem in which x_1 takes on a positive value in the solution). This is the same as saying that the allowable increase in the coefficient is 7.

The reduced cost of a basic variable is always zero (because you need not change the objective function at all to make the variable positive). Neglecting rare cases in which a basis variable takes on the value 0 in a solution, you can figure out reduced costs from the other information in the table: If the final value is positive, then the reduced cost is zero. If the final value is zero, then the reduced cost is negative one times the allowable increase. Remarkably, the reduced cost of a variable is also the amount of slack in the dual constraint associated with the variable. With this interpretation, complementary slackness implies that if a variable that takes on a positive value in the solution, then its reduced cost is zero.

Sensitivity Information on Constraints

The second sensitivity table discusses the constraints. The cell column identifies the location of the left-hand side of a constraint; the name column gives its name (if any); the final value is the value of the left-hand side when you plug in the final values for the variables; the shadow price is the dual variable associated with the constraint; the constraint R.H. side is the right hand side of the constraint; allowable increase tells you by how much you can increase the right-hand side of the constraint without changing the basis; the allowable decrease tells you by how much you can decrease the right-hand side of the constraint without changing the basis.

Complementary Slackness guarantees a relationship between the columns in the constraint table. The difference between the “Constraint Right-Hand Side” column and the “Final Value” column is the slack. (So, from the table, the slack for the three constraints is 0 (= 12- 12), 37 (= 7 + (30)), and 0 (= 10-10), respectively. We know from Complementary Slackness that if there is slack in the constraint then the associated dual variable is zero. Hence CS tells us that the second dual variable must be zero.

Like the case of changes in the variables, you can figure out information on allowable changes from other information in the table. The allowable increase and decrease of non-binding variables can be computed knowing final value and right-hand side constant. If a constraint is not binding, then adding more of the resource is not going to change your solution. Hence the allowable increase of a resource is infinite for a non-binding constraint. (A nearly equivalent, and also true, statement is that the allowable increase of a resource is infinite for a constraint with slack.) In the example, this explains why the allowable increase of the second constraint is infinite. One other quantity is also no surprise. The allowable decrease of a non-binding constraint is equal to the slack in the constraint. Hence the allowable decrease in the second constraint is 37. This means that if you decrease the right-hand side of the second constraint from its original value (7) to anything greater than 30 you do not change the optimal basis. In fact, the only part of the solution that changes when you do this is that the value of the slack variable for this constraint changes. In this paragraph, the point is only this: If you solve an LP and find that a constraint is not binding, then you can remove all of unused (slack) portion of the resource associated with this constraint and not change the solution to the problem.

The allowable increases and decreases for constraints that have no slack are more complicated. Consider the first constraint. The information in the table says that if the right-hand side of the first constraint is between 10 (original value 12 minus allowable decrease 2) and infinity, then the basis of the problem does not change. What these columns do not say is that the solution of the problem does change. Saying that the basis does not change means that the variables that were zero in the original solution continue to be zero in the new problem (with the right-hand side of the constraint changed). However, when the amount of available resource changes, necessarily the values of the other variables change. (You can think about this in many ways. Recall example like the diet problem.

If your diet provides exactly the right amount of Vitamin C, but then for some reason you learn that you need more Vitamin C. You will certainly change what you eat and (if you aren't getting your Vitamin C through pills supplying pure Vitamin C) in order to do so you probably will need to change the composition of your diet a little more of some foods and perhaps less of others. We can say that (within the allowable range) you will not change the foods that you eat in positive amounts. That is, if you ate only spinach and oranges and bagels before, then you will only eat these things (but in different quantities) after the change. Another thing that you can do is simply re-solve the LP with a different right-hand side constant and compare the result.

To finish the discussion, consider the third constraint in the example. The values for the allowable increase and allowable decrease guarantee that the basis that is optimal for the original problem (when the right-hand side of the third constraint is equal to 10) remains obtain provided that the right-hand side constant in this constraint is between -2.3333 and 12.

Here is a way to think about this range. Suppose that your LP involves four production processes and uses three basic ingredients. Call the ingredients land, labor, and capital. The outputs vary use different combinations of the ingredients. Maybe they are growing fruit (using lots of land and labor), cleaning bathrooms (using lots of labor), making cars (using lots of labor and a bit of capital), and making computers (using lots of capital). For the initial specification of available resources, you find that your want to grow fruit and make cars. If you get an increase in the amount of capital, you may wish to shift into building computers instead of cars. If you experience a decrease in the amount of capital, you may wish to shift away from building cars and into cleaning bathrooms instead.

As always, when dealing with duality relationships, the “Adjustable Cells” table and the “Constraints” table really provide the same information. Dual variables correspond to primal constraints. Primal variables correspond to dual constraints. Hence, the “Adjustable Cells” table tells you how sensitive primal variables and dual constraints are to changes in the primal objective function. The “Constraints” table tells you how sensitive dual variables and primal constraints are to changes in the dual objective function (right-hand side constants in the primal).

1.8 Formulation and Solution

In this section, an example of formulation will be presented and then solution and sensitivity results will be discussed.
Example:

Imagine a furniture company that makes tables and chairs. A table requires 40 board feet of wood and a chair requires 30 board feet of wood. Wood costs $1 per board foot and 40,000 board feet of wood are available. It takes 2 hours of skilled labor to make an unfinished table or an unfinished chair. Three more hours of labor will turn an unfinished table into a finished table; two more hours of skilled labor will turn an unfinished chair into a finished chair. There are 6000 hours of skilled labor available. (Assume that you do not need to pay for this labor.) The prices of output are given in the table below:

Product	Price
Unfinished Table	$70
Finished Table	$140
Unfinished Chair	$60
Finished Chair	$110

We want to formulate an LP that describes the production plans that the firm can use to maximize its profits.
The relevant variables are the number of finished and unfinished tables, we will call them TF and TU, and the number of finished and unfinished chairs, CF and CU. The revenue is (using the table):
70 TU + 140 TF + 60C U + 110 CF,

While the cost is 40 TU + 40 TF + 30 CU + 30 CF (because lumber costs $1 per board foot)

The constraints are:
1. 40TU + 40TF + 30CU + 30CF ≤ 40000.
2. 2TU + 5TF + 2CU + 4CF ≤ 6000.

The first constraint says that the amount of lumber used is no more than what is available. The second constraint states that the amount of labor used is no more than what is available.

The answer to the problem to be to construct only finished chairs (1333.333 - I'm not sure what it means to make a sell 1 chair, but let's assume that this is possible). The profit is $106,666.67.

Here are some sensitivity questions.

1. What would happen if the price of unfinished chairs went up? Currently they sell for $60. Because the allowable increase in the coefficient is $50, it would not be profitable to produce them even if they sold for the same amount as finished chairs. If the price of unfinished chairs went down, then certainly you wouldn't change your solution.

2. What would happen if the price of unfinished tables went up? Here something apparently absurd happens. The allowable increase is greater than 70. That is, even if you could sell unfinished tables for more than finished tables, you would not want to sell them. How could this be? The answer is that at current prices you don't want to sell finished tables. Hence it is not enough to make unfinished tables more profitable than finished tables; you must make them more profitable than finished chairs. Doing so requires an even greater increase in the price.

3. What if the price of finished chairs fell to $100? This change would alter your production

plan, since this would involve a $10 decrease in the price of finished chairs and the allowable decrease is only $5. In order to figure out what happens, you need to re-solve the problem. It turns out that the best thing to do is specialize in finished tables, producing 1000 and earning $100,000. Notice that if you continued with the old production plan your profit would be 70 × 13331 = 93, 3331, so the change in production plan was worth more than $6,000.

4. How would profit change if lumber supplies changed? The shadow price of the lumber constraint is $2.67. The range of values for which the basis remains unchanged is 0 to 45,000. This means that if the lumber supply went up by 5000, then you would continue to specialize in finished chairs, and your profit would go up by $2.67 5000 = $10, 333. At this point you presumably run out of labor and want to re-optimize. If lumber supply decreased, then your profit would decrease, but you would still specialize in finished chairs.

5. How much would you be willing to pay an additional carpenter? Skilled labor is not worth anything to you. You are not using the labor than you have. Hence, you would pay nothing for additional workers.

6. Suppose that industrial regulations complicate the finishing process, so that it takes one extra hour per chair or table to turn an unfinished product into a finished one. How would this change your plans?

   You cannot read your answer off the sensitivity table, but a bit of common sense tells you something. The change cannot make you better off. On the other hand, to produce 1,333.33 finished chairs you'll need 1,333.33 extra hours of labor. You do not have that available. So the change will change your profit. It becomes optimal to specialize in finished tables, producing 1000 of them and earning $100,000. (This problem differs from the original one because the amount of labor to create a finished product increases by one unit.)

7. The owner of the firm comes up with a design for a beautiful hand-crafted cabinet. Each cabinet requires 250 hours of labor (this is 6 weeks of full time work) and uses 50 board feet of lumber. Suppose that the company can sell a cabinet for $200, would it be worthwhile?

   You could solve this problem by changing the problem and adding an additional variable and an additional constraint. Note that the coefficient of cabinets in the objective function is 150, which reflects the sale price minus the cost of lumber. Computation has been made. The final value increased to 106,802.7211. The solution involved reducing the output of unfinished chairs to 1319.727891 and increasing the output of cabinets to 8.163265306. (Again, please tolerate the fractions).

You could not have guessed these figures in advance, but you could figure out that making cabinets was a good idea. The way to do this is to value the inputs to the production of cabinets. Cabinets require labor, but labor has a shadow price of zero. They also require lumber. The shadow price of lumber is $2.67, which means that each unit of lumber adds $2.67 to profit. Hence 50 board feet of lumber would reduce profit by $133.50. Since this is less than the price at which you can sell cabinets (minus the cost of lumber), you are better off using your resources to build cabinets. (You can check that the increase in profit associated with making cabinets is $16.50, the added profit per unit, times the number of cabinets that you actually produce).

1.9 Sensitivity Analysis for Objective Function Coefficients Using Excel

In this section, we will see how to perform sensitivity analysis for the changes in objective function coefficients using Excel.

Example:

Solve the following LPP using Excel spreadsheet solver.

Maximize $Z = 3 X_1 + 1 X_2$

Subject to:

$X_1 + 3 X_2 \leq 90$ (Constraint X)

$2 X_1 + 2 X_2 \leq 80$ (Constraint Y)

$2 X_1 + 0 X_2 \leq 60$ (Constraint Z)

$X_1, X_2 \geq 0$

Table 1

	A	B	C	D
1		Input requirements per unit of output		
2	Inputs	X1	X2	Inputs available
3	X	1	3	90
4	Y	2	2	80
5	Z	2	0	60

6	Price per unit	3	1	
7		Model description		
8		Decision variables		
9		X1	X2	
10		30	10	
11	Max. revenues	100		
12	Constraints	Inputs used		Inputs available
13	X	60	≤	90
14	Y	80	≤	80
15	Z	60	≤	60
16				

From table 1, it may be observed that the optimum value of decision variables are $X_1 = 30$ and $X_2 = 10$. The associated maximum revenue is Rs 100/-. Now to see the sensitivity on these optimum output levels we propose to consider 5 different values for the per unit contribution coefficient of the product X_2

Case I

Let the price of X_2 be increased to 2 from 1 in the original problem. Then the revised problem will be as follows:

Maximize $Z = 3 X_1 + 2 X_2$

Subject to:

$X_1 + 3 X_2 \leq 90$ (Constraint X)

$2 X_1 + 2 X_2 \leq 80$ (Constraint Y)

$2 X_1 + 0 X_2 \leq 60$ (Constraint Z)

$X_1, X_2 \geq 0$

Table 2

	A	B	C	D
1		Input requirements per unit of output		
2	Inputs	X1	X2	Inputs available
3	X	1	3	90
4	Y	2	2	80
5	Z	2	0	60

6	Price per unit	3	2	
7		Model description		
8		Decision variables		
9		X1	X2	
10		30	10	
11	Max. revenues	110		
12	Constraints	Inputs used		Inputs available
13	X	60	≤	90
14	Y	80	≤	80
15	Z	60	≤	60
16				

From table 2, it may be observed that the maximum revenue has gone up to Rs 110/- but the value of decision variables do not change. Therefore, we can say that our optimum solution addressed in terms of decision variables is insensitive for such a small change in the price of X_2. The firm will produce again X1 = 30 and X2 = 10 and earn revenue of Rs 110/-. This need not be the case all the time.

Case II

Let the price of X_2 be increased to 3 from 1 in the original problem. Then the revised problem will be as follows:

Maximize $Z = 3 X_1 + 3 X_2$

Subject to:

$X_1 + 3 X_2 \leq 90$ (Constraint X)

$2 X_1 + 2 X_2 \leq 80$ (Constraint Y)

$2 X_1 + 0 X_2 \leq 60$ (Constraint Z)

$X_1, X_2 \geq 0$

Table 3

	A	B	C	D
1		Input requirements per unit of output		
2	Inputs	X1	X2	Inputs available
3	X	1	3	90
4	Y	2	2	80
5	Z	2	0	60

6	Price per unit	3	3	
7		Model description		
8		Decision variables		
9		X1	X2	
10		30	10	
11	Max. revenues	120		
12	Constraints	Inputs used		Inputs available
13	X	60	≤	90
14	Y	80	≤	80
15	Z	60	≤	60
16				

From table 3, it may be observed that the maximum revenue has gone up to Rs 120/- but the value of decision variables do not change. Therefore, we can say that our optimum solution addressed in terms of decision variables is insensitive for such a small change in the price of X_2. The firm will produce again X1 = 30 and X2 = 10 and earn revenue of Rs 120/-. This need not be the case all the time.

Case III

Let the price of X_2 be increased to 5 from 1 in the original problem. Then the revised problem will be as follows:

Maximize $Z = 3X_1 + 5X_2$

Subject to:

$X_1 + 3X_2 \leq 90$ (Constraint X)

$2X_1 + 2X_2 \leq 80$ (Constraint Y)

$2X_1 + 0X_2 \leq 60$ (Constraint Z)

$X_1, X_2 \geq 0$

Table 4

	A	B	C	D
1		Input requirements per unit of output		
2	Inputs	X1	X2	Inputs available
3	X	1	3	90
4	Y	2	2	80
5	Z	2	0	60

6	Price per unit	3	5	
7		Model description		
8		Decision variables		
9		X1	X2	
10		15	25	
11	Max. revenues	170		
12	Constraints	Inputs used		Inputs available
13	X	90	≤	90
14	Y	80	≤	80
15	Z	30	≤	60
16				

From table 4, it may be observed that the maximum revenue has gone up to Rs 170/- as usual but also the value of decision variables has changed to a new level. Therefore, we can say that our optimum solution addressed in terms of decision variables is sensitive for such a change in the price of X_2 from Rs1 to Rs5. So the upper limit for the non-sensitivity of the optimum solution seems to be with price Rs 3 for X_2.

Case IV

Now, to get the lower limit for non-sensitivity of X_2 let us decrease the price X_2 to 0.5 from the original price 1 and work out the problem. The revised problem will be as follows:

Maximize $Z = 3 X_1 + 0.5 X_2$

Subject to:

$X_1 + 3 X_2 \leq 90$ (Constraint X)

$2 X_1 + 2 X_2 \leq 80$ (Constraint Y)

$2 X_1 + 0 X_2 \leq 60$ (Constraint Z)

$X_1, X_2 \geq 0$

Table 5

	A	B	C	D
1		Input requirements per unit of output		
2	Inputs	X1	X2	Inputs available
3	X	1	3	90
4	Y	2	2	80

5	Z	2	0	60
6	Price per unit	3	0.5	
7		Model description		
8		Decision variables		
9		X1	X2	
10		30	10	
11	Max. revenues	95		
12	Constraints	Inputs used		Inputs available
13	X	60	≤	90
14	Y	80	≤	80
15	Z	60	≤	60
16				

From table 5, it may be observed that the maximum revenue has come down to Rs 95/- but the values of decision variables have not changed at all. Therefore, we can say that our optimum solution addressed in terms of decision variables is insensitive for such a change in the price of X_2 from Rs1 to Rs 0.5.

Case V

Now, to get the lower limit for non-sensitivity of X_2 let us decrease the price X_2 to 0.0 from the original price 1 and work out the problem. The revised problem will be as follows:

Maximize $Z = 3 X_1 + 0.0 X_2$

Subject to:

$X_1 + 3 X_2 \leq 90$ (Constraint X)

$2 X_1 + 2 X_2 \leq 80$ (Constraint Y)

$2 X_1 + 0 X_2 \leq 60$ (Constraint Z)

$X_1, X_2 \geq 0$

Table 6

	A	B	C	D
1		Input requirements per unit of output		
2	Inputs	X1	X2	Inputs available
3	X	1	3	90
4	Y	2	2	80
5	Z	2	0	60

6	Price per unit	3	0	
7		Model description		
8		Decision variables		
9		X1	X2	
10		30	0	
11	Max. revenues	90		
12	Constraints	Inputs used		Inputs available
13	X	30	≤	90
14	Y	60	≤	80
15	Z	60	≤	60
16				

From table 6, it may be observed that the maximum revenue has come down to Rs 90/- but the values of decision variables have not changed. Thus this value may be taken as the lower limit for the said coefficient.

As a final result, we would say that original optimum solution in terms of decision variables is insensitive in the range from 0 to 3 for the price X_2. To arrive at this conclusion remember we have made a lot of effort.

Fortunately, spreadsheet solver solution is inclusive of this option also. It gives the lower and upper limit for both coefficients in the objective function. In the final report box, one will have to select "Sensitivity". After this step, as usual click "OK" for the sensitivity table. The solver sensitivity report for the coefficients of the objectivity function is shown (Table-7) as below:

Table- 7

Adjustable Cells						
Cell	Name	Final Value	Reduce Cost	Objective Coefficients	Allowable increase	Allowable decrease
B15	X_1	30	0	3	1E+30	2
C15	X_2	10	0	1	2	1

From the above table one can read both the lower and upper limits for both the coefficients. In the table the entries below the final value column show the optimum solution values for both X_1 and X_2. For this optimum value the sensitivity analysis table shows both allowable increase and allowable decrease from their respective original levels for X_1 and X_2;

Upper limit = Current value + Allowable increase

Lower Limit = Current value – Allowable decrease

In table-8, both lower and upper limits are obtained and reported for both the coefficients.

Table-8

Decision variables	Lower limit	Allowable increase	Current value	Allowable decrease	Upper limit
X_1	1	2	3	1E+30	1E+30
X_2	0	1	1	2	3

Note: In table for X2 the allowable increase is stated as 1E+30 to indicate the infinity. Thus for X2 coefficients the allowable increase is infinity.

1.10 Check Your Progress

1. To solve a linear programming problem with thousands of variables and constraints

   a) personal computer can be used
   b) mainframe computer is required
   c) the problem must be partitioned into subparts
   d) unique software would need to be developed.

2. The amount by which an objective function coefficient can change before a different set of values for the decision variables becomes optimal is the

   a) optimal solution
   b) dual solution
   c) range of optimality
   d) range of feasibility

3. The measure that compares the marginal contribution of a variable with the marginal worth of the resources it consumes is the

   a) shadow price
   b) allowable decrease
   c) allowable increase

d) reduced cost

4. Surplus refers to the

a) LHS value of a ≥ constraint
b) difference between the LHS and RHS values of a ≤ constraint
c) RHS value of a ≥ constraint
d) difference between the LHS and RHS values of a ≥ constraint.

5. If the objective function coefficient of a variable changes within its allowable range

a) the current variable values and the objective value change
b) the current variable values remain the same, but the objective value changes
c) the current variable values and the objective value remain the same
d) none

1.11 Summary

Sensitivity Analysis deals with finding out the amount by which we can change the input data for the output of our linear programming model to remain comparatively unchanged. This helps us in determining the sensitivity of the data we supply for the problem. It is the study of how the uncertainty in the output of a mathematical model or system (numerical or otherwise) can be divided and allocated to different sources of uncertainty in its inputs. It may be understood as an increased understanding of the relationships between input and output variables in a system or model. This helps us in determining the sensitivity of the data we supply for the problem. If a small change in the input (for example in the change in the availability of some raw material) produces a large change in the optimal solution for some model, and a corresponding small change in the input for some other model doesn't affect its optimal solution as much, we can conclude that the second problem is more robust then the first. The second model is less sensitive to the changes in the input data

Sensitivity analysis in excel helps us study the uncertainty in the output of the model with the changes in the input variables. It primarily does stress testing of our modeled assumptions and leads to value-added insights.

1.12 Keywords

Optimum Solution: The solution where the objective function is minimized or maximized.

Shadow Price: Price of a resource is the change in the optimum value of the objective function per unit increase of the resource.

Sensitivity Analysis: The term sensitivity analysis, sometimes also called post-optimality analysis, refers to an analysis of the effect on the optimal solution of changes in the parameters of problem on the current optimal solution

1.13 Self Assessment Test:

Q1.For the LPP given below, conduct the sensitivity analysis for the contribution coefficient of the objective function.

Maximize $Z = 10 X_1 + 15 X_2$

Subject to:

$X_1 + 2 X_2 \leq 70$ (Labour)

$4 X_1 + 2 X_2 \leq 100$ (Capital)

$X_1, X_2 \geq 0$

Q2.For the LPP given below, conduct the sensitivity analysis for the Right Hand Side variables.

Maximize $Z = 45 X_1 + 80 X_2$

Subject to:

$5 X_1 + 20 X_2 \leq 400$ (constraint 1)

$10 X_1 + 15 X_2 \leq 450$ (constraint 2)

$X_1, X_2 \geq 0$

Q3.Solve the following LPP and conduct the sensitivity analysis for the resource coefficients:

Maximize $Z = 3 X_1 + 3 X_2 + 4 X_3 + 1 X_4$

Subject to:

$X_1 + X_2 + X_3 + X_4 = 6$

$5 X_1 + 3 X_2 + 2 X_3 + X_4 = 60$

$X_1 + 3 X_2 + 5 X_3 + 6 X_4 = 50$

$X_1, X_2, X_3, X_4 \geq 0$

1.14 Answers to check your progress

1. **a)** personal computer can be used
2. c) range of optimality
3. d) reduced cost
4. d) difference between the LHS and RHS values of a ≥ constraint
5. b) the current variable values remain the same, but the objective value changes

1.15 References/ Suggested Readings:

Mustafi, C.K. 1988. Operations Research Methods and Practices, Wiley Eastern Limited, New Delhi. Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

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Levin, R and Kirkpatrick, C.A. 1978. Quantitative Approached to Management, Tata McGraw Hill, Kogakusha Ltd., International Student Edition.

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.:6	Vetter:
Linear Programming Problem (LPP): Transportation Model	

Structure:

1.1 Introduction

1.2 Transportation Algorithm

1.3 Basic Feasible Solution of a Transportation Problem

1.4 Modified Distribution Method

1.5 Unbalanced Transportation Problem

1.6 Degenerate Transportation Problem

1.7 Transshipment Problem

1.8 Transportation Problem Maximization

1.9 Check your progress

1.10 Summary

1.11 Keywords

1.12 Self Assessment Test

1.13 Answers to check your progress

1.14 References/ Suggested Readings

Learning Objectives:

After Studying this lesson, students will be able to:

1.1 Formulation of a Transportation Problem

1.2 Determine basic feasible solution using various methods

1.3 Understand the MODI, Stepping Stone Methods for cost minimization

1.4 Make unbalanced Transportation Problem into balanced one using appropriate method

1.5 Solve Degenerate Problem

1.6 Formulate and Solve Transshipment Problem

1.7 Describe suitable method for maximizing the objective function instead of minimizing

1.1 Introduction

A special class of linear programming problem is Transportation Problem, where the objective is to minimize the cost of distributing a product from a number of sources (e.g. factories) to a number of destinations (e.g. warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given rout is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

The transportation problem special feature is illustrated here with the help of following Example.

Example 1.1:

Suppose a manufacturing company owns three factories (sources) and distribute his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

Retail Agency						
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Usually the above table is referred as Transportation Table, which provides the basic information regarding the transportation problem. The quantities inside the table are known as transportation cost per unit of product. The capacity of the factories 1, 2, 3 is 50, 100 and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100, 60, 50, 50, and 40 respectively.

In this case, the transportation cost of one unit from factory 1 to retail agency 1 is 1, from factory 1 to retail agency 2 is 9, from factory 1 to retail agency 3 is 13, and so on. A transportation problem can be formulated as linear programming problem using variables with two subscripts.

Let

x_{11}=Amount to be transported from factory 1 to retail agency 1 x_{12}= Amount to be transported from factory 1 to retail agency 2

……..

……..

……..

……..

x_{35}= Amount to be transported from factory 3 to retail agency 5.

Let the transportation cost per unit be represented by C_{11}, C_{12}, …..C_{35} that is C_{11}=1, C_{12}=9, and so on.

Let the capacities of the three factories be represented by a_1=50, a_2=100, a_3=150.

Let the requirement of the retail agencies are b_1=100, b_2=60, b_3=50, b_4=50, and b_5=40. Thus, the problem can be formulated as

Minimize

$C_{11}x_{11}+C_{12}x_{12}+\ldots\ldots +C_{35}x_{35}$

Subject to:

$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a_1$

$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a_2$

$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a_3$

$x_{11} + x_{21} + x_{31} = b_1$

$x_{12} + x_{22} + x_{32} = b_2$

$x_{13} + x_{23} + x_{33} = b_3$

$x_{14} + x_{24} + x_{34} = b_4$

$x_{15} + x_{25} + x_{35} = b_5$

$x_{11}, x_{12}, \ldots\ldots, x_{35} \geq 0$.

Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem. There are varieties of procedures, which are described in the next section.

1.2 Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

Step 2: Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.

1.3 Basic Feasible Solution of a Transportation Problem

The computation of an initial feasible solution is illustrated in this section with the help of the example1.1 discussed in the previous section. The problem in the example 1.1 has 8 constraints and 15 variables we can eliminate one of the constraints since $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4 + b_5$. Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most 7 non-zero X_{ij}. Generally, any basic feasible solution with m sources (such as factories) and n destination (such as retail agency) has at most m + n -1 non-zero X_{ij}.

The special structure of the transportation problem allows securing a non artificial basic feasible solution using one the following three methods.

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

The difference among these three methods is the quality of the initial basic feasible solution they produce, in the sense that a better that a better initial solution yields a smaller objective value. Generally the Vogel Approximation Method produces the best initial basic feasible solution, and the North West Corner Method produces the worst, but the North West Corner Method involves least computations.

North West Corner Method:

The method starts at the North West (upper left) corner cell of the tableau (variable x_{11}).

Step -1: Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

Step -2: Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

Step -3: If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step -1.

Example 1.2:

Consider the problem discussed in Example 1.1 to illustrate the North West Corner Method of determining basic feasible solution.

Retail Agency						
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

The allocation is shown in the following tableau:

	1	2	3	4	5	Capacity
1	1 **50**	9	13	36	51	~~50~~
2	24 **50**	12 **50**	16	20	1	~~100~~ ~~50~~
3	14	33 **10**	1 **50**	23 **50**	26 **40**	~~150~~ ~~140~~ ~~90~~ ~~40~~
Requirement	~~100~~ ~~50~~	~~60~~ ~~10~~	~~50~~	~~50~~	~~40~~	

The arrows show the order in which the allocated (bolded) amounts are generated. The starting basic solution is given as

$x_{11} = 50$,
$x_{21} = 50$, $x_{22} = 50$
$x_{32} = 10$, $x_{33} = 50$, $x_{34} = 50$, $x_{35} = 40$
The corresponding transportation cost is

50 * 1 + 50 * 24 + 50 * 12 + 10 * 33 + 50 * 1 + 50 * 23 + 40 * 26 = 4420

It is clear that as soon as a value of Xij is determined, a row (column) is eliminated from further consideration. The last value of Xij eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most m + n – 1 positive Xij if the transportation problem has m sources and n destinations.

Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which C_{ij} is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly. If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

Example 1.3:

The least cost method of determining initial basic feasible solution is illustrated with the help of problem presented in the section 1.1.

									Capacity
	1		9		13		10	51	~~50~~
50									
	24	**60**	12		16		20	**40** 1	~~50 60~~
	14		33		1		23	25	~~150 100 50~~
50				**50**		**50**			
Requirement	~~100~~ ~~50~~	~~60~~		~~50~~		~~50~~		~~40~~	

The Least Cost method is applied in the following manner:

We observe that C_{11}=1 is the minimum unit cost in the table. Hence X_{11}=50 and the first row is crossed out since the row has no more capacity. Then the minimum unit cost in the uncrossed-out row and column is C_{25}=1, hence X_{25}=40 and the fifth column is crossed out. Next C_{33}=1is the minimum unit cost, hence X_{33}=50 and the third column is crossed out. Next C_{22}=12 is the minimum unit cost, hence X_{22}=60 and the second column is crossed out. Next we look for the uncrossed-out row and column now C_{31}=14 is the minimum unit cost, hence X_{31}=50 and crossed out the first column since it was satisfied. Finally C_{34}=23 is the minimum unit cost, hence X_{34}=50 and the fourth column is crossed out.

So that the basic feasible solution developed by the Least Cost Method has transportation cost is 1 * 50 + 12 * 60 + 1 * 40 + 14 * 50 + 1 * 50 + 23 * 50 = 2710

Note that the minimum transportation cost obtained by the least cost method is much lower than the corresponding cost of the solution developed by using the north-west corner method.

Vogel Approximation Method (VAM):

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

Step 1: For each row (column) with strictly positive capacity (requirement), determine a penalty by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column).

Step 2: Identify the row or column with the largest penalty among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

Step 3: We select X_{ij} as a basic variable if C_{ij} is the minimum cost in the row or column with largest penalty. We choose the numerical value of X_{ij} as high as possible subject to the row and the column constraints. Depending upon whether a_i or b_j is the smaller of the two i^{th} row or j^{th} column is crossed out.

Step 4: The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

Example 1.4:

Consider the following transportation problem

Origin	Destination				a_i
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240

Note: a_i = capacity (supply) b_j = requirement (demand)

Now, compute the penalty for various rows and columns which is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

Look for the highest penalty in the row or column, the highest penalty occurs in the second column and the minimum unit cost i.e. c_{ij} in this column is c_{12}=22. Hence assign 40 to this cell i.e. x12 = 40 and cross out the second column (since second column was satisfied. This is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4	80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	**4**	**15**	**8**	**3**		

The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the first row and the minimum unit cost in this row is c_{14}=4, hence x_{14}=80 and cross out the first row.

The modified table is as follows:

The next highest penalty in the uncrossed-out rows and columns is 8 which occurs in the third column and the minimum cost in this column is c23=9, hence x23=30 and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				a_i Column Penalty
	1	2	3	4	
1	20	22 **40**	17	4 **80**	0
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240
Row Penalty	4	15	8	3	

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7	40	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the second row and the smallest cost in this row is c_{24}=15, hence x_{24}=30 and cross out the fourth column with the adjusted capacity, requirement and penalty values. The modified table is as follows:

z Origin	Destinat				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7 **30**	10	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	

Row Penalty	8	15	8	8	

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the second row and the smallest cost in this row is c_{21}=24, hence x_{i21}=10 and cross out the second row with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination 1	2	3	4	a_i	Column Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24 **10**	37	9 **30**	7 **30**	0	17
3	32	37	20	15	50	17
b_j	60	40	30	11	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the third row and the smallest cost in this row is c_{31}=32, hence x_{i31}=50 and cross out the third row or first column. The modified table is as follows:

Origin	Destina 1	2	3	4	a_i	Column Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24 **10**	37	9 **30**	7 **30**	0	17
3	32 **50**	37	20	15	0	17
b_j	60	40	30	11	240	
RowPenalty	8	15	8	8		

The transportation cost corresponding to this choice of basic variables is 22 * 40 + 4 * 80 + 9 * 30 + 7 * 30 + 24 * 10 + 32 * 50 = 3520

1.4 Modified Distribution Method

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method.

Step 1: Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

Step 2: Introduce dual variables corresponding to the row constraints and the column constraints. If there are m origins and n destinations then there will be m+n dual variables. The dual variables corresponding to the row constraints are represented by $u_{i,}$ i=1,2,…..m where as the dual variables corresponding to the column constraints are represented by v_j, j=1,2,…..n. The values of the dual variables are calculated from the equation given below

$u_i + v_j = c_{ij}$ if $x_{ij} > 0$

Step 3: Any basic feasible solution has m + n -1 $x_{ij} > 0$. Thus, there will be m + n -1 equation to determine m + n dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

Step 4: If $x_{ij}=0$, the dual variables calculated in Step 3 are compared with the c_{ij} values of this allocation as $c_{ij} – u_i – v_j$. If a_l $c_{ij} – u_i – vj \geq 0$, then by the theorem of complementary slackness it can be shown that the corresponding solution of the transportation problem is optimum. If one or more $c_{ij} – u_i – v_j < 0$, we select the cell with the least value of $c_{ij} – u_i – v_j$ and allocate as much as possible subject to the row and column constraints. The allocations of the number of adjacent cell are adjusted so that a basic variable becomes non-basic.

Step 5: A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

Example 1.5:

For example consider the transportation problem given below:

						Supply
	1	9	13	36	51	50
	24	12	16	20	1	100
	14	33	1	23	26	150
Demand	100	70	50	40	40	

Step 1:

First we have to determine the basic feasible solution. The basic feasible solution using least cost method is

x_{11}=50, x_{22}=60, x_{25}=40, x_{31}=50, x_{32}=10, x_{33}=50 and x_{34}=40

Step 2:

The dual variables u_1, u_2, u_3 and v_1, v_2, v_3, v_4, v_5 can be calculated from the corresponding cij values, that is

$u_1+v_1=1$ $u_2+v_2=12$ $u_2+v_5=1$ $u_3+v_1=14$ $u_3+v_2=33$

$u_3+v_3=1$ $u_3+v_4=23$

Step 3:

Choose one of the dual variables arbitrarily is zero that is $u_3= 0$ as it occurs most often in the above equations. The values of the variables calculated are

$u_1= -13$, $u_2= -21$, $u_3=0$

$v_1=14$, $v_2=33$, $v_3=1$, $v_4=23$, $v_5=22$

Step 4:

Now we calculate $c_{ij} – u_i – v_j$ values for all the cells where $x_{ij}=0$ (.e. unallocated cell by the basic feasible solution)

That is

Cell(1,2)= $c_{12}-u_1-v_2 = 9+13-33 = -11$

Cell(1,3)= $c_{13}-u_1-v_3 = 13+13-1 = 25$

Cell(1,4)= $c_{14}-u_1-v_4 = 36+13-23 = 26$

Cell(1,5)= $c_{15}-u_1-v_5 = 51+13-22 = 42$

Cell(2,1)= $c_{21}-u_2-v_1 = 24+21-14 = 31$

Cell(2,3)= $c_{23}-u_2-v_3 = 16+21-1 = 36$

Cell(2,4)= $c_{24}-u_2-v_4 = 20+21-23 = 18$

Cell(3,5)= $c_{35}-u_3-v_5 = 26-0-22 = 4$

Note that in the above calculation all the $c_{ij} – u_i – v_j \geq 0$ except for cell (1, 2) where $c_{12} – u_1 – v_2 = 9+13- 33 = -11$.

Thus in the next iteration x12 will be a basic variable changing one of the present basic variables non-basic. We also observe that for allocating one unit in cell (1, 2) we have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net transportation cost for each unit of such reallocation is

-33 -1 + 9 +14 = -11

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in the cell (3, 2) will be negative. Thus, the revised basic feasible solution is

$x_{11} = 40$, $x_{12} = 10$, $x_{22} = 60$, $x_{25} = 40$, $x_{31} = 60$, $x_{33} = 50$, $x_{34} = 40$

1.5 Unbalanced Transportation Problem

In the previous section we discussed about the balanced transportation problem i.e. the total supply (capacity) at the origins is equal to the total demand (requirement) at the destination. In this section we are going to discuss about the unbalanced transportation problems i.e. when the total supply is not equal to the total demand, which are called as unbalanced transportation problem.

In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost.

Example 1.6:

Consider the following unbalanced transportation problem

			Warehouse		Supply
		W1	W2	W3	
Plant	X	20	17	25	400
	Y	10	10	20	500
Demand		400	400	500	

In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

			Warehouse		Supply
		W1	W2	W3	
Plant	X	20	17	25	400
	Y	10	10	20	500
Unsatisfied Demand		0	0	0	400
Demand		400	400	500	

Now we can solve as balanced problem discussed as in the previous sections.

1.6 Degenerate Transportation Problem

In a transportation problem, if a basic feasible solution with m origins and n destinations has less than m + n -1 positive X_{ij} i.e. occupied cells, then the problem is said to be a degenerate transportation problem. The degeneracy problem does not cause any serious difficulty, but it can cause computational problem wile determining the optimal minimum solution.

Therefore, it is important to identify a degenerate problem as early as beginning and take the necessary action to avoid any computational difficulty. The degeneracy can be identified through the following results:

“In a transportation problem, a degenerate basic feasible solution exists if and only if some partial sum of supply (row) is equal to a partial sum of demand (column). For example the following transportation problem is degenerate. Because in this problem

$a_1 = 400 = b_1$

$a_2 + a_3 = 900 = b_2 + b_3$

			Warehouse		Supply
		W1	W2	W3	
Plant	X	20	17	25	400
	Y	10	10	20	500
Unsat.Demand		0	0	0	400
Demand		400	400	500	

There is a technique called perturbation, which helps to solve the degenerate problems.

Perturbation Technique:

The degeneracy of the transportation problem can be avoided if we ensure that no partial sum of ai (supply) and bj (demand) is equal. We set up a new problem where

$a_i = a_i + d$ i = 1, 2, ……, m

$b_j = b_j$ j = 1, 2, ……, n -1

$b_n = b_n + m_d$ d > 0

This modified problem is constructed in such a way that no partial sum of a_i is equal to the b_j. Once the problem is solved, we substitute d = 0 leading to optimum solution of the original problem.

Example 1.7:

Consider the following problem

		Warehouse			Supply
		W1	W2	W3	
Plant	X	20	17	25	400 + d
	Y	10	10	20	500 + d
Unsat.Demand		0	0	0	400 + d
Demand		400	400	500 +3d	1300 + 3d

Now this modified problem can be solved by using any of the three methods viz. North-west Corner, Least Cost, or VAM.

1.7 Transshipment Problem

There could be a situation where it might be more economical to transport consignments in several sages that is initially within certain origins and destinations and finally to the ultimate receipt points, instead of transporting the consignments from an origin to a destination as in the transportation problem.

The movement of consignment involves two different modes of transport viz. road and railways or between stations connected by metre gauge and broad gauge lines. Similarly it is not uncommon to maintain dumps for central storage of certain bulk material. These require transshipment.

Thus for the purpose of transshipment the distinction between an origin and destination is dropped so that from a transportation problem with m origins and n destinations we obtain a transshipment problem with m + n origins and m + n destinations.

The formulation and solution of a transshipment problem is illustrated with the following Example.

Example 1.8:

Consider the following transportation problem where the origins are plants and destinations are depots.

		Depot			**Supply**
		W1	W2	W3	
Plant	A	1	3	15	**150**
	B	3	5	25	**300**
Demand		**150**	**150**	**150**	**450**

When each plant is also considered as a destination and each depot is also considered as an origin, there are altogether five origins and five destinations. So that some additional cost data are necessary, they are as follows:

Unit transportation cost From Plant To Plant to

		Plant	
		A	B
Plant	A	0	55
	B	2	0

Unit transportation cost From Depot To Depot
To

		Depot		
		X	Y	Z
Depot	X	0	25	2
	Y	2	0	3
	Z	55	3	0

		Plant	
		A	B
Depot	X	3	15
	Y	25	3
	Z	45	55

Now, from the above Tables, we obtain the transportation formulation of the transshipment problem, which is shown in the following table

Transshipment Table

	A	B	X	Y	Z	Supply
A	0	55	1	3	15	150+450=600
B	2	0	3	5	25	300+450=750
X	3	15	0	25	2	450
Y	25	3	2	0	3	450
Z	45	55	55	3	0	450
Demand	450	450	150+450=600	150+450=600	150+450=600	

A buffer stock of 450 which is the total supply and total demand in the original transportation problem is added to each row and column of the transshipment problem. The resulting transportation problem has m + n = 5 origins and m + n = 5 destinations.

By solving the transportation problem presented in the above transshipment table, we obtain

x_{11}=150 x_{13}=300 x_{14}=150 x_{21}=3001 x_{22}=450 x_{33}=300
x_{35}=150 x_{44}=450 x_{55}=450

The transshipment problem explanation is as follows:

1. Transport x21=300 from plant B to plant A. This increase the availability at plant A to 450 units including the 150 originally available from A.
2. From plant A transport x13=300 to depot X and x14=150 to depot Y.
3. From depot X transport x35=150 to depot Z. Thus, the total cost of transshipment is:

2*300 + 3 * 150 + 1*300 + 2*150 = $1650

Note: The consignments are transported from pants A, B to depots X, Y, Z only according to the transportation Table 1, the minimum transportation cost schedule is x_{13}=150 x_{21}=150 x_{22}=150 with a minimum cost of 3450.

Thus, transshipment reduces the cost of consignment movement.

1.8 Transportation Problem Maximization

There are certain types of transportation problem where the objective function is to be maximized instead of minimized. These kinds of problems can be solved by converting the maximization problem into minimization problem. The conversion of maximization into minimization is done by subtracting the unit costs from the highest unit cost of the table.
The maximization of transportation problem is illustrated with the following Example 1.8.

Example 1.8:

A company has three factories located in three cities viz. X, Y, Z. These factories supplies consignments to four dealers viz. A, B, C and D. The dealers are spread all over the country. The production capacity of these factories is 1000, 700 and 900 units per month respectively. The net return per unit product is given in the following table.

		Dealer				Capacity
		A	B	C	D	
Factory	X	6	6	6	4	1000
	Y	4	2	4	5	700
	Z	5	6	7	8	900

Requirement	900	800	500	400	2600

Determine a suitable allocation to maximize the total return.

This is a maximization problem. Hence first we have to convert this in to minimization problem. The conversion of maximization into minimization is done by subtracting the unit cost of the table from the highest unit cost.

Look the table, here 8 is the highest unit cost. So, subtract all the unit cost from the 8, and then we get the revised minimization transportation table, which is given below.

		Dealer				Capacity
		A	B	C	D	
Factory	X	2	2	2	4	$1000=a_1$
	Y	4	6	4	3	$700=a_2$
	Z	3	2	1	0	$900=a_3$
Requirement		$900=b_1$	$800=b_2$	$500=b_3$	$400=b_4$	2600

Now we can solve the problem as a minimization problem.

The problem here is degenerate, since the partial sum of $a_1=b_2+b_3$ or $a_3=b_3$. So consider the corresponding perturbed problem, which is shown below.

		Dealer				Capacity
		A	B	C	D	
Factory	X	2	2	2	4	1000+d
	Y	4	6	4	3	700+d
	Z	3	2	1	0	900+d
Requirement		900	800	500	400 + 3d	2600

First we have to find out the basic feasible solution. The basic feasible solution by lest cost method is $x_{11}=100+d$, $x_{22}=700-d$, $x_{23}=2d$, $x_{33}=500-2d$ and $x_{34}=400+3d$.

Once if the basic feasible solution is found, next we have to determine the optimum solution using MODI (Modified Distribution Method) method. By using this method we obtain

$u_1+v_1=2$ $u_1+v_2=2$ $u_2+v_2=6$

$u_2+v_3=4$ $u_3+v_3=1$ $u_3+v_4=0$

Taking $u_1=0$ arbitrarily we obtain

$u_1=0$, $u_2=4$, $u_3=1$ and $v_1=2$, $v_2=3$, $v_3=0$

On verifying the condition of optimality, we know that

C_{12}-u_1-v_2 < 0   and C_{32}-u_3-v_2 <0

So, we allocate x_{12}=700 - d and make readjustment in some of the other basic variables. The revised values are:

x_{11}=200+d, x_{12}=800, x_{21}=700 - d, x_{23}=2d, x_{33}=500-3d, and x_{34}=400+3d

$u_1+v_1=2$ $u_1+v_2=2$ $u_2+v_1=4$ $u_2+v_3=4$ $u_3+v_3=1$ $u_3+v_4=0$

Taking u_1=0 arbitrarily we obtain

u_1=0, u_2=2, u_3=-1 v_1=2, v_2=2, v_3=2, v_4=1

Now, the optimality condition is satisfied.

Finally, taking d=0 the optimum solution of the transportation problem is X_{11}=200, x_{12}=800, x_{21}=700, x_{33}=500 and x_{34}=400

Thus, the maximum return is:

6*200 + 6*800 + 4*700 + 7*500 + 8*400 = 15500

1.8 Check your progress:

1. The Enfield Company is very concerned about shipping costs when moving its products from its various factories to its regional warehouses. To minimize these costs, it should use
   a. the transportation model
   b. the assignment model
   c. the shortest-path model
   d. the maximal-flow model.

2. The initial solution to a transportation problem can be generated in any manner, so long as
   a. it ignores cost
   b. all supply and demand are satisfied
   c. degeneracy does not exist
   d. all cells are filled

3. A transportation problem has a feasible solution when
   a. all of the improvement indexes are positive

b. the number of filled cells is one less than the number of rows plus the number of columns

c. the solution yields the lowest possible cost

d. all demand and supply constraints are satisfied

4. The total cost of the optimal solution to a transportation problem

a. is calculated by multiplying the total supply (including any dummy values) by the average cost of the cells

b. cannot be calculated from the information given

c. can be calculated from the original non-optimal cost, by adding the savings made at each improvement

d. can be calculated based only on the entries in the filled cells of the solution

5. Which of the following statements about the northwest corner rule is false?

a. One must exhaust the supply for each row before moving down to the next row

b. One must exhaust the demand requirements of each column before moving to the next column

c. When moving to a new row or column, one must select the cell with the lowest cost

d. One must check that all supply and demand constraints are met.

1.10 Summary

Transportation Problem is a special kind of linear programming problem. Because of the transportation problem special structure the simplex method is not suitable. But which may be utilized to make efficient computational techniques for its solution.

Generally transportation problem has a number of origins and destination. A certain amount of consignment is available in each origin. Similarly, each destination has a certain demand/requirements. The transportation problem represents amount of consignment to be transported from different origins to destinations so that the transportation cost is minimized with out violating the supply and demand constraints.

There are two phases in the transportation problem. First is the determination of basic feasible solution and second is the determination of optimum solution.

There are three methods available to determine the basic feasible solution, they are

1. North West Corner Method
2. Least Cost Method or Matrix Minimum Method
3. Vogel's Approximation Method (VAM)

In order to determine optimum solution we can use either one of the following method

1. Modified Distribution (MODI) Method

Or

2. Stepping Stone Method

Transportation problem can be generalized into a Transshipment Problem where transportation of consignment is possible from origin to origin or destination as well as destination to origin or destination. The transshipment problem may be result in an economy way of shipping in some situations.

1.11 Keywords

Origin: is the location from which the shipments are dispatched.

Destination: is the location to which the shipments are transported.

Unit Transportation Cost: is the transportation cost per unit from an origin to destination. Perturbation Technique: is a method of modifying a degenerate transportation problem in order to solve the degeneracy.

1.12 Self Assessment Test

Q1. Four companies viz. W, X, Y and Z supply the requirements of three warehouses viz. A, B and C respectively. The companies' availability, warehouses requirements and the unit cost of transportation are given in the following table. Find an initial basic feasible solution using

a. North West Corner Method
b. Least Cost Method
c. Vogel Approximation Method (VAM)

	Warehouses			
Company	A	B	C	**Supply**
W	10	8	9	15
X	5	2	3	20
Y	6	7	4	30
Z	7	6	9	35

Requirement	25	26	49	100

Q2. Find the optimum Solution of the following Problem using MODI method.

		Dealer			Capacity
		A	B	C	
Factory	X	8	9	10	42
	Y	9	11	11	30
	Z	10	12	9	28
Requirement		35	40	35	100

Q3. The ABT transport company ships truckloads of food grains from three sources viz. X, Y, Z to four mills viz. A, B, C, D respectively. The supply and the demand together with the unit transportation cost per truckload on the different routes are described in the following transportation table. Assume that the unit transportation costs are in hundreds of dollars. Determine the optimum minimum shipment cost of transportation using MODI method.

		Mill				
Source		A	B	C	D	**Supply**
	X	10	2	20	11	15
	Y	12	7	9	20	25
	Z	4	14	16	18	10
Demand		5	15	15	15	

Q4. An organization has three plants at X, Y, Z which supply to warehouses located at A, B, C, D, and E respectively. The capacity of the plants is 800, 500 and 900 per month and the requirement of the warehouses is 400, 400, 500, 400 and 800 units respectively. The following table shows the unit transportation cost.

	A	B	C	D	E
X	5	8	6	6	3
Y	4	7	7	6	6
Z	8	4	6	6	3

Q5. Solve the following transshipment problem

Consider a transportation problem has two sources and three depots. The availability, requirements and unit cost are as follows:

		Destinations			
		A	B	C	Availability
Sources	X	9	8	1	30
	Y	1	7	8	30
	Requirement	20	20	20	60

In addition to the above, suppose that the unit cost of transportation from source to source and from depot to depot are as:

		S1	S2
Sources	S1	9	8
	S2	1	7

		D1	D2	D3
Depot	D1	0	2	1
	D2	2	0	9
	D3	1	9	0

Find out minimum transshipment cost of the problem and also compare this cost with the corresponding minimum transportation cost.

Q6. Sharma Garments, Jaipur is interested to purchase the following type and quantities of dresses

Type of dress	V	W	X	Y	Z
Quantity	150	100	75	250	200

Four different dress makers are submitted the tenders, who undertake to supply not more than the quantities indicated below:

Type of dress	A	B	C	D
Quantity	300	250	150	200

Sharma Garments estimates that its profit per dress will vary according to the dress maker as indicates in the following table:

	V	W	X	Y	Z
A	2.75	3.5	4.25	2.25	1.5
B	3	3.25	4.5	1.75	1
C	2.5	3.5	4.75	2	1.25
D	3.25	2.75	4	2.5	1.75

Determine how should the orders to be places for the dresses so as to maximize the profit.

1.13 Answers to check your progress

1. a) transportation model
2. b) all supply and demand are satisfied
3. d) all demand and supply constraints are satisfied
4. d) can be calculated based only on the entries in the filled cells of the solution
5. c) When moving to a new row or column, one must select the cell with the lowest cost

1.14 References/ Suggested Readings

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.: 7	Vetter:
Assignment Model	

Structure

1.1 Introduction

1.2 Assignment Problem Structure and Solution

1.3 Unbalanced Assignment Problem

1.4 Infeasible Assignment Problem

1.5 Maximization in an Assignment Problem

1.6 Crew Assignment Problem

1.7 Check your progress

1.8 Summary

1.9 Keywords

1.10 Self Assessment Test

1.11 Answers to check your progress

1.12 Further References

Learning Objectives

After Studying this lesson, students will be able to:

- ❖ Assignment Problem Formulation
- ❖ How to solve the Assignment Problem
- ❖ How to solve the unbalanced problem using appropriate Method
- ❖ Make appropriate modification when some problems are infeasible
- ❖ Modify the problem when the objective is to maximize the objective function
- ❖ Formulate and solve the crew assignment problems

1.1 Introduction

The Assignment Problem can define as follows:

Given n facilities, n jobs and the effectiveness of each facility to each job, here the problem is to assign each facility to one and only one job so that the measure of effectiveness if optimized. Here the optimization means Maximized or Minimized. There are many management problems has a assignment problem structure. For example, the head of the department may have 6 people available for assignment and 6 jobs to fill. Here the head may like to know which job should be assigned to which person so that all tasks can be accomplished in the shortest time possible. Another example a container company may have an empty container in each of the location 1, 2,3,4,5 and requires an empty container in each of the locations 6, 7, 8,9,10. It would like to ascertain the assignments of containers to various locations so as to minimize the total distance.

The third example here is, a marketing set up by making an estimate of sales performance for different salesmen as well as for different cities one could assign a particular salesman to a particular city with a view to maximize the overall sales.

Note that with n facilities and n jobs there are n! possible assignments. The simplest way of finding an optimum assignment is to write all the n! possible arrangements, evaluate their total cost and select the assignment with minimum cost. Bust this method leads to a calculation problem of formidable size even when the value of n is moderate. For n=10 the possible number of arrangements is 3268800.

1.2 Assignment Problem Structure and Solution

The structure of the Assignment problem is similar to a transportation problem, is as follows:

Jobs

		1	2	…	n	
	1	c_{11}	c_{12}	…	c_{1n}	1
	2	c_{21}	c_{21}	…	c_{2n}	1
		.	.	..	⋱	.
Workers		.	.	.	.	
	n	c_{n1}	c_{n2}	…	c_{nn}	1
		1	1	…1		

The element c_{ij} represents the measure of effectiveness when i^{th} person is assigned j^{th} job. Assume that the overall measure of effectiveness is to be minimized. The element x_{ij} represents the number of i^{th} individuals assigned to the j^{th} job. Since i^{th} person can be assigned only one job and j^{th} job can be assigned to only one person we have the following

$x_{i1} + x_{i2} + \ldots\ldots\ldots\ldots + x_{in} = 1$, where i = 1, 2,.. , n

$x_{1j} + x_{2j} + \ldots\ldots\ldots\ldots + x_{nj} = 1$, where j = 1, 2,..., n

and the objective function is

formulated as Minimize $c_{11}x_{11}$ +

$c_{12}x_{12}$ + ... + $c_{nn}x_{nn}$

$x_{ij} \geq 0$

The assignment problem is actually a special case of the transportation problem where m = n and $a_i = b_j = 1$. However, it may be easily noted that any basic feasible solution of an assignment problem contains $(2_n – 1)$ variables of which (n – 1) variables are zero. Because of this high degree of degeneracy the usual computation techniques of a transportation problem become very inefficient. So, hat a separate computation technique is necessary for the assignment problem.

The solution of the assignment problem is based on the following results:

"If a constant is added to every element of a row/column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original assignment problem and vice versa". – This result may be used in two different methods to solve the assignment problem. If in an assignment problem some cost elements c_{ij} are negative, we may have to convert them into an equivalent assignment problem where all the cost elements are non-negative by adding a suitable large constant to the cost elements of the relevant row or column, and then we look for a feasible solution which has zero assignment cost after adding suitable constants to the cost elements of the various rows and columns. Since it has been assumed that all the cost elements are non-negative, this assignment must be optimum. On the basis of this principle a computational technique known as Hungarian Method is developed. The Hungarian Method is discussed as follows.

Hungarian Method:

The Hungarian Method is discussed in the form of a series of computational steps as follows, when the objective function is that of minimization type.

Step 1:
From the given problem, find out the cost table. Note that if the number of origins is not equal to the number of destinations then a dummy origin or destination must be added.

Step 2:
In each row of the table find out the smallest cost element, subtract this smallest cost element from each element in that row. So, that there will be at least one zero in each row of the new table. This new table is known as First Reduced Cost Table.

Step 3:
In each column of the table find out the smallest cost element, subtract this smallest cost element from each element in that column. As a result of this, each row and column has at least one zero element. This new table is known as Second Reduced Cost Table.

Step 4:
Now determine an assignment as follows:

For each row or column with a single zero element cell that has not be assigned or eliminated, box that zero element as an assigned cell.

1. For every zero that becomes assigned, cross out all other zeros in the same row and for column.
2. If for a row and for a column there are two or more zero and one can't be chosen by inspection, choose the assigned zero cell arbitrarily.
3. The above procedures may be repeated until every zero element cell is either assigned (boxed) or crossed out.

Step 5:
An optimum assignment is found, if the number of assigned cells is equal to the number of rows (and columns). In case we had chosen a zero cell arbitrarily, there may be an alternate optimum. If no optimum solution is found i.e. some rows or columns without an assignment then go to **Step 6**.

Step 6:
Draw a set of lines equal to the number of assignments which has been made in **Step 4**, covering all the zeros in the following manner

1. Mark check (√) to those rows where no assignment has been made.
2. Examine the checked (√) rows. If any zero element cell occurs in those rows, check (√) the respective columns that contains those zeros.
3. Examine the checked (√) columns. If any assigned zero element occurs in those columns, check (√) the respective rows that contain those assigned zeros.
4. The process may be repeated until now more rows or column can be checked.
5. Draw lines through all unchecked rows and through all checked columns.

Step 7:

Examine those elements that are not covered by a line. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them.

Add this smallest element to every element that lies at the intersection of two lines. Then the resulting matrix is a new revised cost table

Example 2.1:

Problem

A work shop contains four persons available for work on the four jobs. Only one person can work on any one job. The following table shows the cost of assigning each person to each job. The objective is to assign person to jobs such that the total assignment cost is a minimum.

Jobs

		1	**2**	**3**	**4**
Persons	**A**	20	25	22	28
	B	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

Solution

As per the Hungarian Method

Step 1: The cost Table

Jobs

		1	**2**	**3**	**4**
Persons	**A**	20	25	22	28
	B	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

Step 2: Find the First Reduced Cost Table

		Jobs			
		1	**2**	**3**	**4**
	A	0	5	2	8
Persons	**B**	0	3	8	2
	C	2	0	4	7
	D	2	0	1	1

Step 3: Find the Second Reduced Cost Table

		Jobs			
		1	**2**	**3**	**4**
	A	0	5	1	7
Persons	**B**	0	3	7	1
	C	2	0	3	6
	D	2	0	0	0

Step 4: Determine an Assignment

By examine row A of the table in Step 3, we find that it has only one zero (cell A1) box this zero and cross out all other zeros in the boxed column. In this way we can eliminate cell B1.

Now examine row C, we find that it has one zero (cell C2) box this zero and cross out (eliminate) the zeros in the boxed column. This is how cell D2 gets eliminated.

There is one zero in the column 3. Therefore, cell D3 gets boxed and this enables us to eliminate cell D4.

Therefore, we can box (assign) or cross out (eliminate) all zeros. The resultant table is shown below:

Jobs

		1	2	3	4
	A	[0]	5	1	7
Persons	B	~~0~~	3	7	1
	C	2	[0]	3	6
	D	2	~~0~~	[0]	~~0~~

Step 5:

The solution obtained in Step 4 is not optimal. Because we were able to make three assignments when four were required.

Step 6:

Cover all the zeros of the table shown in the Step 4 with three lines (since already we made three assignments).

Check row B since it has no assignment. Note that row B has a zero in column 1, therefore check column1. Then we check row A since it has a zero in column 1. Note that no other rows and columns are checked. Now we may draw three lines through unchecked rows (row C and D) and the checked column (column 1). This is shown in the table given below:

Jobs

	1	2	3	4
Persons	[0]	5	1	7
	0	3	7	1
	~~2~~	[0]	3	6
	2	0		0

Step 7:
Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (C1 and D1) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

		Jobs			
		1	**2**	**3**	**4**
	A	0	4	0	6
Persons	**B**	0	2	6	0
	C	3	0	3	6
	D	3	0	0	0

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:
Determine an assignment

Examine each of the four rows in the table given in Step 7, we may find that it is only row C which has only one zero box this cell C2 and cross out D2.

Note that all the remaining rows and columns have two zeros. Choose a zero arbitrarily, say A1 and box this cell so that the cells A3 and B1 get eliminated.

Now row B (cell B4) and column 3 (cell D4) has one zero box these cells so that cell D4 is eliminated.

Thus, all the zeros are either boxed or eliminated. This is shown in the following table

		Jobs 1	2	3	4
Persons	A	[0]	4	~~0~~	6
	B	0	2	6	[0]
	C	~~3~~	[0]	3	6
	D	3	~~0~~	[0]	~~0~~

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

The total cost of assignment is: 78 that is A1 + B4 + C2 + D3

20 + 17 + 17 + 24 = 78

1.3 Unbalanced Assignment Problem

In the previous section we assumed that the number of persons to be assigned and the number of jobs were same. Such kind of assignment problem is called as **balanced assignment problem**. Suppose if the number of person is different from the number of jobs then the assignment problem is called as **unbalanced**.

If the number of jobs is less than the number of persons, some of them can't be assigned any job. So that we have to introduce on or more dummy jobs of zero duration to make the unbalanced assignment problem into balanced assignment problem. This balanced assignment problem can be solved by using the Hungarian Method as discussed in the previous section. The persons to whom the dummy jobs are assigned are left out of assignment.

Similarly, if the number of persons is less than number of jobs then we have introduce one or more dummy persons with zero duration to modify the unbalanced into balanced and then the problem is solved using the Hungarian Method. Here the jobs assigned to the dummy persons are left out.

Example 2.2:

Solve the following unbalanced assignment problem of minimizing the total time for performing all the jobs.

		Jobs 1	2	3	4	5
Workers	**A**	5	2	4	2	5
	B	2	4	7	6	6
	C	6	7	5	8	7
	D	5	2	3	3	4
	E	8	3	7	8	6
	F	3	6	3	5	7

		Jobs					
		1	**2**	**3**	**4**	**5**	**6**
	A	5	2	4	2	5	0
	B	2	4	7	6	6	0
Workers	**C**	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Now the problem becomes balanced one since the number of workers is equal to the number jobs. So that the problem can be solved using Hungarian Method.

Step 1: The cost Table

		Jobs					
		1	**2**	**3**	**4**	**5**	**6**
	A	5	2	4	2	5	0
Workers	**B**	2	4	7	6	6	0
	C	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Step 2: Find the First Reduced Cost Table

		Jobs 1	2	3	4	5	6
	A	5	2	4	2	5	0
	B	2	4	7	6	6	0
Workers	**C**	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Step 3: Find the Second Reduced Cost Table

		Jobs 1	2	3	4	5	6
	A	3	0	1	0	1	0
	B	0	2	4	4	2	0
Workers	**C**	4	5	2	6	3	0
	D	3	0	0	1	0	0
	E	6	1	4	6	2	0
		1	4	0	3	3	0

Step 4: Determine an Assignment

By using the Hungarian Method the assignment is made as follows:

Jobs

Workers		**1**	**2**	**3**	**4**	**5**	**6**
	A	3	~~0~~	1	[0]	1	~~0~~
	B	~~0~~	2	4	4	2	~~0~~
		[0]	5	2	6	3	×
	D	3	0	0	1	[0]	~~0~~
	E	6	1	4	6	2	[0]
	F	1	4	[0]	3	3	~~0~~

Step 5:

The solution obtained in Step 4 is not optimal. Because we were able to make five assignments when six were required.

Step 6:

Cover all the zeros of the table shown in the Step 4 with five lines (since already we made five assignments).

Check row E since it has no assignment. No

That row B has a zero in column 6, therefore check column6. Then we check row C since it has a zero in column 6. Note that no other rows and columns are checked. Now we may draw five lines through unchecked rows (row A, B, D and F) and the checked column (column 6). This is shown in the table given below:

		Jobs					
		1	**2**	**3**	**4**	**5**	**6**
Workers	**A**	3	0	1	0	1	0
	B		2	4	4	2	0
	C	4 0	5	2	6	3	
	D	3	0	0	1	0	0
	E	6	1	4	6	2	0 0
	F	1	4	0	3	3	0

Step 7:
Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element

from the uncovered cells and add 1 to elements (A6, B6, D6 and F6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

Jobs

		1	2	3	4	5	6
	A	3	0	1	0	1	1
	B	0	2	4	4	2	1
Workers	**C**	3	4	1	5	2	0
	D	3	0	0	1	0	1
	E	5	0	3	5	1	0
	F	1	4	0	3	3	1

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:

Determine an assignment

Jobs

		1	2	3	4	5	6
	A	3	~~0~~	1	[0]	1	1
	B	[0]	2	4	4	2	1
Workers	**C**	3	4	1	5	2	[0]
	D	3	~~0~~	~~0~~	1	[0]	1
	E	5	[0]	3	5	1	~~0~~
	F	1	4	[0]	3	3	1

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

Thus, the worker A is assigned to Job4, worker B is assigned to job 1, worker C is assigned to job 6, worker D is assigned to job 5, worker E is assigned to job 2, and worker F is assigned to job 3. Since the Job 6 is dummy so that worker C can't be assigned.

The total minimum time is: 14 that is A4 + B1 + D5 + E2 + F3

2 + 2 + 4 + 3 + 3 = 14

Example 2.3:

A marketing company wants to assign three employees viz. A, B, and C to four offices located at W, X, Y and Z respectively. The assignment cost for this purpose is given in following table.

		Offices			
		W	**X**	**Y**	**Z**
Employees	**A**	160	220	240	200
	B	100	320	260	160
	C	100	200	460	250

Solution

Since the problem has fewer employees than offices so that we have introduce a dummy employee with zero cost of assignment.

The revised problem is as follows:

		Offices			
		W	**X**	**Y**	**Z**
	A	160	220	240	200
Employees	**B**	100	320	260	160
	C	100	200	460	250
	D	0	0	0	0

Now the problem becomes balanced. This can be solved by using Hungarian Method as in the case of Example 2.2. Thus as per the Hungarian Method the assignment made as follows: Employee A is assigned to Office X, Employee B is assigned to Office Z, Employee C is assigned to Office W and Employee D is assigned to Office Y. Note that D is empty so that no one is assigned to Office Y.

The minimum cost of assignment is: 220 + 160 + 100 = 480

1.4 Infeasible Assignment Problem

Sometimes it is possible a particular person is incapable of performing certain job or a specific job can't be performed on a particular machine. In this case the solution of the problem takes into account of these restrictions so that the infeasible assignment can be avoided.

The infeasible assignment can be avoided by assigning a very high cost to the cells where assignments are restricted or prohibited. This is explained in the following Example 2.4.

Example 2.4:

A computer centre has five jobs to be done and has five computer machines to perform them. The cost of processing of each job on any machine is shown in the table below.

Computer Machines	Jobs 1	2	3	4	5
1	70	30	X	60	30
2	X	70	50	30	30
3	60	X	50	70	60
4	60	70	20	40	X
5	30	30	40	X	70

Because of specific job requirement and machine configurations certain jobs can't be done on certain machines. These have been shown by X in the cost table. The assignment of jobs to the machines must be done on a one to one basis. The objective here is to assign the jobs to the available machines so as to minimize the total cost without violating the restrictions as mentioned above.

Solution

Step 1: The cost Table

Because certain jobs cannot be done on certain machines we assign a high cost say for example 500 to these cells i.e. cells with X and modify the cost table. The revised assignment problem is as follows:

Computer Machines	Jobs 1	2	3	4	5
1	70	30	500	60	30
2	500	70	50	30	30
3	60	500	50	70	60
4	60	70	20	40	500
5	30	30	40	500	70

Now we can solve this problem using Hungarian Method as discussed in the previous sections.

Step 2: Find the First Reduced Cost Table

		Jobs				
		1	**2**	**3**	**4**	**5**
Computer Machines	**1**	40	0	470	30	0
	2	470	40	20	0	0
	3	10	450	0	20	10
	4	40	50	0	20	480
	5	0	0	10	470	40

Step 3: Find the Second Reduced Cost Table

		Jobs				
		1	**2**	**3**	**4**	**5**
Computer Machines	**1**	40	0	470	30	0
	2	470	40	20	0	0
	3	10	450	0	20	10
	4	40	50	0	20	480
	5	0	0	10	470	40

Step 4: Determine an Assignment

		Jobs				
		1	**2**	**3**	**4**	**5**
	1	40	[0]	470	30	~~0~~
Computer Machines	**2**	470	40	20	0	0
	3	10	450	~~0~~	[0]	~~10~~
	4	40	50	[0]	20	480
	5	[0]	~~0~~	~~0~~	470	40

Step 5:

The solution obtained in Step 4 is not optimal. Because we were able to make four assignments when five were required.

Step 6:

Cover all the zeros of the table

Shown in the Step 4 with four lines (since already we made four assignments)

Check row 4 since it has no assignment. Note that row 4 has a zero in column 3, therefore check column3. Then we check row 3 since it has a zero in column 3. Note that no other rows and columns are checked. Now we may draw four lines through unchecked rows (row 1, 2, 3 and 5) and the checked column (column 3).

This is shown in the table given below:

Jobs

Computer Machines	1	2	3	4	5
1	40	0	470	30	0
2	470	40	20		0
3	10	450	0	0	10
4	40	50	0	20	480
5	0	0	0	470	40

Step 7:
Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 10. Subtract this smallest element from the uncovered cells and add 1 to elements (A6, B6, D6 and F6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

Jobs

Computer Machines	1	2	3	4	5
	0				
1	40	0	471	30	0
2	470	40	21	0	0
3	0	440	0	10	0
4	30	40	0	10	470
5	0	0	11	470	40

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:

Determine an assignment

		Jobs 1	2	3	4	5
Computer Machines	**1**	40	~~0~~	471	30	[0]
	2	470	40	21	[0]	~~0~~
	3	[0]	440	~~0~~	~~0~~	~~0~~
	4	30	40	[0]	10	470
	5	~~0~~	[0]	11	470	40

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

Thus, the Machine1 is assigned to Job5, Machine 2 is assigned to job4, Machine3 is assigned to job1, Machine4 is assigned to job3 and Machine5 is assigned to job2.

The minimum assignment cost is: 170

1.5 Maximization in an Assignment Problem

There are situations where certain facilities have to be assigned to a number of jobs so as to maximize the overall performance of the assignment. In such cases the problem can be converted into a minimization problem and can be solved by using Hungarian Method. Here the conversion of maximization problem into a minimization can be done by subtracting all the elements of the cost table from the highest value of that table.

Example 2.5:

Consider the problem of five different machines can do any of the required five jobs with different profits resulting from each assignment as illustrated below:

Jobs		Machines 1	2	3	4	5
	1	40	47	50	38	50
	2	50	34	37	31	46
	3	50	42	43	40	45
	4	35	48	50	46	46
	5	38	72	51	51	49

Find out the maximum profit through optimal assignment.

Solution

This is a maximization problem, so that first we have to find out the highest value in the table and subtract all the values from the highest value. In this case the highest value is 72.

The new revised table is given below:

Jobs		Machines 1	2	3	4	5
	1	32	35	22	34	22
	2	22	38	35	41	26
	3	22	30	29	32	27
	4	37	24	22	26	26
	5	34	0	21	21	23

This can be solved by using the Hungarian Method.

By solving this, we obtain the solution is as follows:

Jobs	Machines
1	3
2	5
3	1
4	4
5	2

The maximum profit through this assignment is: 264

1.6 Crew Assignment Problem

The crew assignment problem is explained with the help of the following problem

Problem:

A trip from Chennai to Coimbatore takes six hours by bus. A typical time table of the bus service in both the direction is given in the Table 1. The cost of providing this service by the company based on the time spent by the bus crew i.e. driver and conductor away from their places in addition to service times. The company has five crews. The condition here is that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours for the return trip. Also the company has guest house facilities for the crew of Chennai as well as at Coimbatore.

Find which line of service is connected with which other line so as to reduce the waiting time to the minimum.

Table 1

Departure from Chennai	Route Number	Arrival at Coimbatore	Arrival at Chennai	Route Number	Departure from Coimbatore
06.00	1	12.00	11.30	a	05.30
07.30	2	13.30	15.00	b	09.00
11.30	3	17.30	21.00	c	15.00
19.00	4	01.00	00.30	d	18.30
00.30	5	06.30	06.00	e	00.00

Solution

For each line the service time is constant so that it does not include directly in the computation.

Suppose if the entire crew resides at Chennai then the waiting times in hours at Coimbatore for different route connections are given below in Table 2.

If route 1 is combined with route a, the crew after arriving at Coimbatore at 12 Noon start at 5.30 next morning. Thus the waiting time is 17.5 hours. Some of the assignments are infeasible. Route c leaves Coimbatore at 15.00 hours. Thus the crew of route 1 reaching Coimbatore at 12 Noon are unable to take the minimum stipulated rest of four hours if they are asked to leave by route c. Hence 1-c is an infeasible assignment. Thus it cost is M (a large positive number).

Table 2

Route	**a**	**B**	**c**	**d**	**e**
1	17.5	21	M	6.5	12
2	16	19.5	M	5	10.5
3	12	15.5	21.5	M	6.5
4	4.5	8	4	17.5	23
5	23	M	8.5	12	17.5

Similarly, if the crews are assumed to reside at Coimbatore then the waiting times of the crew in hours at Chennai for different route combinations are given below in Table 3.

Table 3

Route	a	B	c	d	e
1	18.5	15	9	5.5	M
2	20	16.5	10.5	7	M
3	M	20.5	14.5	11	5.5
4	7.5	M	22	18.5	13
5	13	9.5	M	M	18.5

Suppose, if the crew can be instructed to reside either at Chennai or at Coimbatore, minimum waiting time from the above operation can be computed for different route combination by choosing the minimum of the two waiting times (shown in the Table 2 and Table 3). This is given in the following Table 4.

Table 4

Route	a	b	c	d	e
1	17.5*	15	9	5.5	12*
2	16*	16.5	10.5	5*	10.5*
3	12*	15.5*	14.5	11	5.5
4	4.5*	8*	14*	17.5*	13
5	13	9.5	8.5*	12*	17.5*

Note: The asterisk marked waiting times denotes that the crew are based at Chennai; otherwise they are based at Coimbatore.

Now we can solve the assignment problem (presented in Table 4) using Hungarian Method.

Step 1: Cost Table (Table 5)

Table 5

Route	a	b	c	d	e
1	17.5*	15	9	5.5	12*
2	16*	16.5	10.5	5*	10.5*
3	12*	15.5*	14.5	11	5.5
4	4.5*	8*	14*	17.5*	13
5	13	9.5	8.5*	12*	17.5*

Step 2: Find the First Reduced cost table (Table 6)

Table 6

Route	a	B	c	d	e
1	12	9.5	3.5	0	6.5
2	11	11.5	5.5	0	5.5
3	6.5	10	9	5.5	0
4	0	3.5	9.5	13	8.5
5	4.5	1	0	3.5	9

Step 3: Find the Second Reduced cost table (Table 7)

Table 7

Route	A	B	c	d	e
1	12	8.5	3.5	0	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	5.5	0
4	0	2.5	9.5	13	8.5
5	4.5	0	0	3.5	9

Step 4: Determine an Assignment (Table 8)

Table 8

Route	A	B	C	d	e
1	12	8.5	3.5	[0]	6.5
2	11	10.5	5.5	~~0~~	5.5
3	6.5	9	9	5.5	[0]
4	[0]	2.5	9.5	13	8.5
5	4.5	0	0	3.5	9

Step 5: The solution obtained in Step 4 is not optimal since the number of assignments are less than the number of rows (columns).

Step 6: Check (√) row 2 since it has no assignment and note that row 2 has a zero in column d, therefore check (√) column d also. Then check row 1 since it has zero in column d. Draw the lines through the unchecked rows and checked column using 4 lines (only 4 assignments are made). This is shown in Table 9.

Table 9

Route	**a**	**b**	**C**	**d**	**e**
1	12	8.5	3.5	[0]	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	~~5.5~~	[0]
4	[0]	2.5	9.5	13	8.5
5	4.5	[0]	0	3.5	9 [0]

Step 7: Develop a new revised table (Table 10)

Take the smallest element from the elements not covered by the lines in this case 3.5 is the smallest element. Subtract all the uncovered elements from 3.5 and add 3.5 to the elements lie at the intersection of two lines (cells 3d, 4d and 5d). The new revised table is presented in Table 10.

Table 10

Route	a	B	C	d	E
1	8.5	5	0	0	3
2	7.5	7	2	0	2
3	6.5	9	9	9	0
4	0	2.5	9.5	16.5	8.5
5	4.5	0	0	7	9

Step 8: Go to Step 4 and repeat the procedure until an optimal solution if arrived.

Step 9: Determine an Assignment (Table 11)

Table 11

Route	a	b	c	d	e
1	8.5	5	[0]	~~0~~	3
2	7.5	7	2	[0]	2
3	6.5	9	9	~~0~~	[0]
4	[0]	2.5	9.5	16.5	8.5
5	4.5	[0]	~~0~~	7	9

Assignment illustrated in Table 11 is **optimal** since the **number of assignments is equal to the number of rows (columns).**

Thus, the routes to be prepared to achieve the minimum waiting time are as follow

1-c, 2 – d, 3 – e, 4 – a and 5 – b

By referring Table 5, we can obtain the waiting times of these assignments as well as the residence (guest house) of the crews. This is presented in the following Table 12.

Table 12

Routes	Residence of the Crew	Waiting Time
1 – c	Coimbatore	9
2 – d	Chennai	5
3 – e	Coimbatore	5.5
4 – a	Chennai	4.5
5 - b	Coimbatore	9.5

1.7 Check Your Progress

You are required to do some activities to check your progress. Answer the followings:

1. An assignment problem is considered as a special case of transportation problem because

   a. Number of rows equals columns
   b. All X_{ij} = 0 or 1
   c. All rim conditions are 1
   d. All of the above

2. Which method usually gives a very good solution to the assignment problem?

   a. northwest corner rule
   b. Vogel's approximation method
   c. MODI method
   d. stepping-stone method
   e. none of the above

3. An assignment problem can be viewed as a special case of transportation problem in which the capacity from each source is ________ and the demand at each destination is _________.

a. 1; 1
b. Infinity; infinity
c. 1000; 1000
d. -1; -1

4. __________ occurs when the number of occupied squares is less than the number of rows plus the number of columns minus one.

   a. Degeneracy
   b. Infeasibility
   c. Unboundedness
   d. Unbalance

5. Both transportation and assignment problems are members of a category of LP problems called ______.

   a. shipping problems
   b. logistics problems
   c. network flow problems
   d. routing problems

1.8 Summary

Assignment model is a specially structured linear programming problem that is of minimization nature like transportation model. It may also be used in maximization objective. The assignment problem is used for the allocation of a number of persons to a number of jobs so that the total time of completion is minimized. The assignment problem is said to be **balanced** if it has equal number of person and jobs to be assigned. If the number of persons (jobs) is different from the number of jobs (persons) then the problem is said to be **unbalanced**. An unbalanced assignment problem can be solved by converting into a balanced assignment problem. The conversion is done by introducing dummy person or a dummy job with zero cost.

Because of the special structure of the assignment problem, it is solved by using a special method known as **Hungarian Method**.

1.9 Keywords

Cost Table: The completion time or cost corresponding to every assignment is written down in a table form if referred as a cost table.

Hungarian Method: is a technique of solving assignment problems.

Assignment Problem: is a special kind of linear programming problem where the objective is to minimize the assignment cost or time.

Balanced Assignment Problem: is an assignment problem where the number of persons equal to the number of jobs.

Unbalanced Assignment Problem: is an assignment problem where the number of jobs is not equal to the number of persons.

***Infeasible Assignment Problem*:** is an assignment problem where a particular person is unable to perform a particular job or certain job cannot be done by certain machines.

1.10 Self Assessment Test

Q1. Solve the following assignment problem

	1	2	3	4	5
1	Rs.3	Rs.8	Rs.2	Rs.10	Rs.3
2	Rs.8	Rs.7	Rs.2	Rs.9	Rs.7
3	Rs.6	Rs.4	Rs.2	Rs.7	Rs.5
4	Rs.8	Rs.4	Rs.2	Rs.3	Rs.5
5	Rs.9	Rs.10	Rs.6	Rs.9	Rs.10

Q2. Work out the various steps of the solution of the Example 2.3.

Q3. A steel company has five jobs to be done and has five softening machines to do them. The cost of softening each job on any machine is given in the following cost matrix. The assignment of jobs to machines must be done on a one to one basis. Here is the objective is to assign the jobs to the machines so as to minimize the total assignment cost without violating the restrictions.

		Jobs				
		1	**2**	**3**	**4**	**5**
Softening Machines	**1**	80	30	X	70	30
	2	70	X	60	40	30
	3	X	80	60	80	70
	4	70	80	30	50	X
	5	30	30	50	X	80

Q4. Work out the various steps of the solution of the problem presented in Example 2.5.

Q5. A marketing manager wants to assign salesman to four cities. He has four salesmen of varying experience. The possible profit for each salesman in each city is given in the following table. Find out an assignment which maximizes the profit.

		Cities			
		1	**2**	**3**	**4**
Salesperson	1	25	27	28	38
	2	28	34	29	40
	3	35	24	32	33
	4	24	32	25	28

Q6. Shiva's three wife, Rani, Brinda, and Fathima want to earn some money to take care of personal expenses during a school trip to the local beach. Mr. Shiva has chosen three chores for his wife:

washing, cooking, sweeping the cars. Mr. Shiva asked them to submit bids for what they feel was a fair pay for each of the three chores. The three wife of Shiva accept his decision. The following table summarizes the bid received.

		Chores		
		Washing 1	**Cooking 2**	**Sweeping 3**
	Rani	25	18	17
Wife's	**Brinda**	17	25	15
	Fathima	18	22	32

Q7. Solve the following problem

		Office			
		O1	**O2**	**O3**	**O4**
	E1	2600	3200	3400	3000
Employees	**E2**	2000	4200	3600	2600
	E3	2000	3000	5600	4000

Q8. The railway operates seven days a week has a time table shown in the following table. Crews (Driver and Guard) must have minimum rest of six hours between trans. Prepare the combination of trains that minimizes waiting time away from the city. Note that for any given combination the crew will be based at the city that results in the smaller waiting time and also find out for each combination the city where the crew should be based at.

Train No.	Departure at Bangalore	Arrival at Chennai	Train No.	Departure at Chennai	Arrival at Bangalore
101	7 AM	9 AM	201	9 AM	11 AM
102	9 AM	11 AM	202	10 AM	12 Noon
103	1.30 PM	3.30 PM	203	3.30 PM	5.30 PM
104	7.30 PM	9.30 PM	204	8 PM	10 PM

1.11Answers to check your progress

1. d) All of the above
2. b) Vogel's approximation method
3. a) 1; 1
4. a) Degeneracy
5. c) network flow problems

1.12 References/Suggested Teadings

Hamdy A Taha, 1999. Introduction to Operations Research, PHI Limited, New Delhi.

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.: 8	Vetter:
Queuing Model	

Structure

1.1 Introduction

1.2 Terminologies of queuing system

1.3 Rules of Queue

1.4 List of Variables to be Used in Queuing Mode

1.5 Traffic intensity (or utilization factor)

1.6 Classification of Queuing models

1.7 Explanation of Empirical Queuing Models

1.8 Check your progress

1.9 Summary

1.10 Key Words

1.11 Self Assessment Test

1.12 Answers to check your progress

1.13 References- Suggested Readings

Learning Objectives

After Studying this lesson, students will be able to:

- ❖ Understand the nature and scope of Queuing Models;
- ❖ Develop better understanding of empirical queuing models
- ❖ Describe the basic queuing system configurations
- ❖ Understand the assumptions of the common models
- ❖ Analyze a variety of operating characteristics of waiting lines

1.1 Introduction:

A flow of customers from finite or infinite population towards the service facility forms a ***queue (waiting line)*** an account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, ***waiting time*** is required either for the service facilities or for the customers arrival. In general, the ***queuing system*** consists of one or more queues and one or more servers and operates under a set of procedures. Depending upon the server status, the incoming customer either waits at the queue or gets the turn to be served. If the server is free at the time of arrival of a customer, the customer can directly enter into the counter for getting service and then leave the system. In this process, over a period of time, the system may experience " Customer waiting" and /or "Server idle time"

Queuing System:

A queuing system can be completely described by

(1) the input (arrival pattern)
(2) the service mechanism (service pattern)
(3) The queue discipline and
(4) Customer's behaviour

1.2 Terminologies of Queuing System

Various terminologies to be used in queuing system have been explained as below-

The input (arrival pattern)

The input described the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random manner which is not possible for prediction. Thus the arrival pattern can be described in terms of probabilities and consequently the probability distribution for ***inter-arrival*** times (the time between two successive arrivals) must be defined. We deal with those Queuing system in which the customers arrive in poisson process. The mean arrival rate is denoted by
λ.

The Service Mechanism

This means the arrangement of service facility to serve customers. If there is infinite number of servers, then all the customers are served instantaneously or arrival and there will be no queue. If the number of servers is finite then the customers are served according to a specific order with service time a constant or a random variable. Distribution of service time follows '**Exponential distribution'** defined by

$$f(t) = \lambda e^{-\lambda t}, \; t > 0$$

The mean Service rate is $E(t) = 1/\lambda$

1.3 Rules of Queue

Queuing Discipline

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are

1. First come first served – (FCFS)
2. First in first out – (FIFO)
3. Last in first out – (LIFO)
4. Selection for service in random order (SIRO)

Customer's behaviour

A. Generally, it is assumed that the customers arrive into the system one by one. But in some cases, customers may arrive in groups. Such arrival is called **Bulk arrival**

B. If there is more than one queue, the customers from one queue may be tempted to join another queue because of its smaller size. This behaviour of customers is known as **jockeying**

C. If the queue length appears very large to a customer, he/she may not join the queue. This property is known as **Balking** of customers

D. Sometimes, a customer who is already in a queue will leave the queue in anticipation of longer waiting line. This kind of departure is known as **Reneging.**

1.4 List of Variables to be used in Queuing Model

The list of variable used in queuing models is give below:

n - No of customers in the system

C - No of servers in the system

P_n (t) – Probability of having n customers in the system at time t

P_n - Steady state probability of having customers in the system

P_0 - Probability of having zero customers in the system

L_q - Average number of customers waiting in the queue

L_s - Average number of customers waiting in the system (in the queue and in the service counters)

W_q - Average waiting time of customers in the queue.

W_s - Average waiting time of customers in the system (in the queue and in the service counters)

δ - Arrival rate of customers

μ - Service rate of server

ϕ - Utilization factor of the server

δ eff - Effective rate of arrival of customers

M - Poisson distribution

N - Maximum numbers of customers permitted in the system. Also, it denotes the size of the calling source of the customers.

GD - General discipline for service. This may be first in first – serve (FIFS), last-in-first serve (LIFS) random order (Ro) etc.

1.5 Traffic intensity (or utilization factor)

An important measure of a simple queue is its traffic intensity given by

$$\text{Traffic intensity } \phi = \frac{\text{Mean arrival time}}{\text{Mean service time}} = \frac{\delta}{\mu} \quad (< 1)$$

and the unit of traffic intensity is Erlang

1.6 Classification of Queuing models

Generally, queuing models can be classified into six categories using Kendall's notation with six parameters to define a model. The parameters of this notation are

P- Arrival rate distribution ie probability law for the arrival /inter – arrival time.

Q - Service rate distribution, ie probability law according to which the customers are being served.

R - Number of Servers (ie number of service stations)

X - Service discipline

Y - Maximum number of customers permitted in the system.

Z - Size of the calling source of the customers.

A queuing model with the above parameters is written as (P/Q/R : X/Y/Z)

Model 1: (M/M/1): (GD/ ∞ / ∞) Model

In this model

a) the arrival rate follows poisson (M) distribution
b) Service rate follows poisson distribution (M)
c) Number of servers is 1
d) Service discipline is general disciple (ie GD)
e) Maximum number of customers permitted in the system is infinite (∞)
f) Size of the calling source is infinite (∞)

The steady state equations to obtain, P_n the probability of having customers in the system and the values for Ls, Lq, Ws and Wq are given below.

$n = 0,1,2,---- \infty$ where $\phi = \delta < \frac{1}{\mu}$

Ls – Average number of customers waiting in the system (ie waiting in the queue and in the service station)

$$P_n = \phi^n (1-\phi)$$

$$Ls = \frac{\phi}{1-\phi}$$

$$Lq = Ls - \frac{\delta}{\mu}$$

$$= \frac{\phi}{1-\phi} - \phi$$

$$= \frac{\phi - (1-\phi)\phi}{1-\phi} = \frac{\phi^2}{1-\phi}$$

Average waiting time of customer
(in the queue and in the service station) $= W_s$

$$Ws = \frac{L_s}{\delta} = \frac{\phi}{(1-\phi)\delta}$$

$$= \frac{\phi}{1-\phi} \times \frac{1}{\mu\phi} \quad \frac{1}{\mu(1-\phi)}$$

(Since $\delta = \mu\phi$)

$$Ws = \frac{1}{\mu - \mu\phi}$$

$$= \frac{1}{\mu - \delta}$$

$$W_s = \frac{1}{\mu - \delta}$$

W_q = Average waiting time of customers in the queue.

$$= L_q / \delta = [1 / \delta] [\phi^2 / [1-\phi]]$$

$$= 1 / \mu\phi [\phi^2 / [1-\phi]]$$

$$= \frac{\phi}{\mu - \mu\phi} \quad \text{(Since } \mu\phi = \delta\text{)}$$

$$Wq = \frac{\phi}{\mu - \delta}$$

1.7 Explanation of Empirical Queuing Model:

The above discussed queuing models have been thoroughly discussed below with the help of few examples:

Example 1:

The arrival rate of customers at a banking counter follows a poisson distibution with a mean of 30 per hours. The service rate of the counter clerk also follows poisson distribution with mean of 45 per hour.

a) What is the probability of having zero customers in the system?
b) What is the probability of having 8 customers in the system?
c) What is the probability of having 12 customers in the system?

d) Find Ls, Lq, Ws and Wq

Solution

Given arrival rate follows poisson distribution with mean =30

$\therefore \delta = 30$ per hour

Given service rate follows poisson distribution with mean = 45

$\therefore \mu = 45$ Per hour

$\therefore$ Utilization factor $\phi = \delta/\mu$

$= 30/45$

$= 2/3$

$= 0.67$

a) The probability of having zero customer in

the system $P_0 = \phi^0 (1-\phi)$

$= 1-\phi$

$= 1-0.67$

$= 0.33$

b) The probability of having 8 customers in

the system $P_8 = \phi^8 (1-\phi)$

$= (0.67)^8 (1-0.67)$

$= 0.0406 \times 0.33$

$= 0.0134$

c) Probability of having 12 customers in the system is

$P_{12} = \phi^{12} (1-\phi)$

$= (0.67)^{12} (1-0.67)$

$= 0.0082 \times 0.33$

= 0.002706

$L_s = \frac{\phi}{1-\phi} = \frac{0.67}{1-0.67}$

$= \frac{0.67}{0.33} = 2.03$

= 2 customers

$L_q = \frac{\phi^2}{1-\phi} = \frac{(0.67)^2}{1-0.67} = \frac{0.4489}{0.33}$

$= 1.36$

= 1 Customer

$$W_s = \frac{1}{\mu - \delta} = \frac{1}{45\text{-}30} = \frac{1}{15}$$

= **0.0666 hour**

$$W_q = \frac{\phi}{\mu - \delta} = \frac{0.67}{45\text{-}30} = \frac{0.67}{15}$$

= **0.4467 hours**

Example 2:

At one-man barbar shop, customers arrive according to poisson dist with mean arrival rate of 5 per hour and the hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:

(i) Average number of customers in the shop and the average numbers waiting for a haircut.

(ii) The percentage of time arrival can walk in straight without having to wait.

(iii) The percentage of customers who have to wait before getting into the barber's chair.

Solution:-

Given mean arrival of customer δ = 5/60 =1/12
and mean time for server μ 1/10

$\therefore \phi = \delta / \mu$ = [1/12] x 1 = 10 /12

= **0.833**

(i) Average number of customers in the system (numbers in the queue and in the service station) L_s $= \phi / 1\text{-} \phi$ = 0.83 / 1- 0.83

= 0.83 / 0.17

= 4.88

= 5 Customers

(ii) The percentage of time arrival can walk straight into barber's chair without waiting is Service utilization $= \phi$ %

$= \delta / \mu$ %

= 0.833 x 100

= 83.3 %

(iii) The percentage of customers who have to wait before getting into the barber's chair = (1-ϕ) % (1- 0.833) % = 0.167 x 100 = **16.7%**

Example 3:

Vehicles are passing through a toll gate at the rate of 70 per hour. The average time to pass through the gate is 45 seconds. The arrival rate and service rate follow poisson distibution. There is a complaint that the vehicles wait for a long duration. The authorities are willing to install one more gate to reduce the average time to pass through the toll gate to 35 seconds if the idle time of the toll gate is less than 9% and the average queue length at the gate is more than 8 vehicle, check whether the installation of the second gate is justified?

Solutions:-

Arrival rate of vehicles at the toll gate δ = 70 per hour

Time taken to pass through the gate= 45 Seconds

Service rate μ = $\frac{1 \text{ hours}}{45 \text{ seconds}}$

= 3600/45 = 80

= 80 Vehicles per hour

$\therefore$ Utilization factor $\phi = \delta/\mu$

= 70 / 80

= 0.875

(a) Waiting no. of vehicles in the queue is L_q

$L_q = \phi^2 / 1 - \phi = \frac{(0.875)^2}{1-0.875}$

= $\frac{0.7656}{0.125}$

= **6.125**

= **6 Vehicles**

(b) Revised time taken to pass through the gate = 30 seconds

$\therefore$The new service rate after installation of an additional gate = 1 hour/35 Seconds = 3600/35

= 102.68 Vehicles / hour

$\therefore$ Utilization factor $\phi = \delta / \mu = 70 / 102.86 =$ **0.681**

Percentage of idle time of the gate = $(1-\phi)\%$

= (1-0.681) %

= 0.319 %
= 31.9 = **32 %**

This idle time is not less than 9% which is expected.

Therefore the installation of the second gate is not justified since the average waiting number of vehicles in the queue is more than 8 but the idle time is not less than 32%. Hence idle time is far greater than the number of vehicles waiting in the queue.

1.8 Explanation of Second Model; (M/M/C): (GD/ ∞/∞)Model

The parameters of this model are as follows:

(i) Arrival rate follows poisson distribution

(ii) Service rate follows poisson distribution

(iii) No of servers is C'.

(iv) Service discipline is general discipline.

(v) Maximum number of customers permitted in the system is infinite

Then the steady state equation to obtain the probability of having n customers in the system is $P_n = \frac{\phi^n P_o}{n!}$; $o \le n \le C$

$$= \frac{\phi^n \ P_o}{C^{n-c} \ C!} \quad \text{for } n > c; \text{ Where } \phi / c < 1$$

Where $[\delta / \mu c] < 1$ as $\phi = \delta / \mu$

$$\therefore P_0 = \{[\sum_{n=0}^{C-1} \phi^n/n!] + \phi^c / (c! \, [1 - \phi/c])\}^{-1}$$

where c! = 1 x 2 x 3 x ....up to C

$L_q = [\phi^{c+1} / [c-1! \, (c - \phi)]] \times P_0$
$= (c\phi \, P_c) / (c - \phi)^2$

$L_s = L_q + \phi$ and $W_s = W_q + 1 / \mu$

$W_q = L_q / \delta$

Under special conditions P_o $= 1 - \phi$ and $L_q = \phi^{C+!} / c^2$ Where ϕ <1 and $P_o = (C\text{-}\phi)\ (c – 1)! / c^c$

and $L_q = \phi / (c\text{-}\phi)$, where $\phi / c < 1$

Example 1:

At a central warehouse, vehicles are at the rate of 24 per hour and the arrival rate follows poisson distribution. The unloading time of the vehicles follows exponential distribution and the unloading rate is 18 vehicles per hour. There are 4 unloading crews. Find

(i) P_o and P_3

(ii) L_q, L_s, W_q and W_s

Solution:

Arrival rate $\delta = 24$ per hour

Unloading rate $\mu = 18$ per hour

No. of unloading crews C=4

$\phi = \delta / \mu$ $= 24 / 18$ **=1.33**

(i) $P_0 = \{[\sum_{n=0}^{C-1} \phi^n/n!] + \phi^c / (c!\ [1 - \phi/c])\}^{-1}$

$= \{[\sum_{n=0}^{3} (1.33)^n/n!] + (1.33)^4 /(4!\ [1 - (1.33)/ 4])\}^{-1}$

$= \{ (1.33)^0 / = 0! + (1.33)^1 / 1! + (1.33)^2 / 2! + (1.33)^3 / 3! + (1.33)^4 / 24!\ [1 - (1.33)/ 4] \}^{-1}$

$= [1 + 1.33 + 0.88 + 0.39 + 3.129/16.62]^{-1}$

$= [3.60 + 0.19]^{-1}$ = $[3.79]^{-1}$

= 0.264

We know $P_n = (\phi^n / n!)\ P_o$ for $0 \le n \le c$

$\therefore P_3 = (\phi^3 / 3!)\ P_o$ Since $0 \le 3 \le 4$

$= [(1.33)^3 / 6] \times 0.264$

$= 2.353 \times 0.044$

= 0.1035

(ii) $L_q = \dfrac{\phi^{C+1}}{(C-1)!\,(C-\phi)^2} \times P_0$

$= \dfrac{(1.33)^5}{3! \times (4-1.33)^2} \times 0.264$

$= \dfrac{(4.1616)}{6 \times (2.77)^2} \times 0.264$

$= \dfrac{(4.1616) \times 0.264}{46.0374}$

$= 1.099 / 46.0374$

$= 0.0239$

= 0.0239 Vehicles

$L_s = L_q + \phi = 0.0239 + 1.33$

= 1.3539 Vehicles

$W_q = L_q / \delta = 0.0239 / 24$

= 0.000996 hrs

$W_s = W_q + 1/\mu = 0.000996 + 1/18$

$= 0.000996 + 0.055555$

$= 0.056551$ hours.

Example 2

A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if the people arrive in poisson fashion at the rate of 10 per hour

a) What is the probability of having to wait for service?
b) What is the expected percentage of idle time for each girl?
c) If a customer has to wait, what is the expected length of his waiting time?

Solution:-

$$P_0 = \{[\sum_{n=0}^{C-1} \phi^n/n!] + \phi^c / (c!\,[1 - \phi/c])\}^{-1}$$

Where $\phi = \delta / \mu \therefore$ given arrival rate = 10 per hour

$\delta = 10 / 60 = 1 / 6$ per minute

Service rate = 4 minutes

$\therefore \mu = 1 / 4$ person per minute

Hence, $\phi = \delta / \mu = (1/6) \times 4$

$= 2/3$

$= \mathbf{0.67}$

$$P_0 = \{[\sum_{n=0}^{1} \phi^n/n!] + (0.67)^2/(2!\,[1 - 0.67/2])\}^{-1}$$

$$= [1 + (\phi / 1!)] + 0.4489 / (2 - 0.67)]^{-1}$$

$$= [1 + 0.67 + 0.4489 / (1.33)]^{-1}$$

$$= [1 + 0.67 + 0.34]^{-1}$$

$$= [2.01]^{-1} = \mathbf{1/2}$$

The Probability of having to wait for the service is

P (w > 0)

$$= \frac{\phi^c}{c!\,[1 - \phi/c]} \times P_0$$

$$= \frac{0.67^2 \times (1/2)}{2!\,[1 - 0.67/2]}$$

$$= 0.4489 / 2.66$$

$$= \mathbf{0.168}$$

b) The probability of idle time for each girl is

$= 1 - P(w > 0)$

$= 1 - 1/3$

$= \mathbf{2/3}$

∴ Percentage of time the service remains idle = 67% approximately

c) The expected length of waiting time (w/w>0)

$= 1 / (c\,\mu - \delta)$

$= 1 / [(1/2) - (1/6)]$

$=$ **3 minutes**

Examples 3

A petrol station has two pumps. The service time follows the exponential distribution with mean 4 minutes and cars arrive for service in a poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What proportion of time the pump remains idle?

Solution: Given C = 2

The arrival rate = 10 cars per hour.

$\therefore \delta = 10 / 60 = 1 / 6$ car per minute

Service rate = 4 minute per cars.

i.e $\mu = ¼$ car per minute.

$\phi = \delta / \mu = (1/6) / (1/4)$

$= 2 / 3$

= 0.67

Proportion of time the pumps remain busy = ϕ / c = 0.67 / 2 = 0.33 = 1/3

$\therefore$ The proportion of time, the pumps remain idle

= 1 – proportion of the pumps remain busy

= 1-1 / 3 = 2 / 3

$$P_0 = \left\{\left[\sum_{n=0}^{C-1} \phi^n/n!\right] + \phi^c / (c!\,[1 - \phi/c])\right\}^{-1}$$

$= [\,(0.67)^0 / 0!) + (0.67)^1 / 1!) + (0.67)^2 / 2!)\,[1- (0.67 / 2)^1\,]^{-1}$

$= [1 + 0.67 + 0.4489 / (1.33)]^{-1}$

$= [1 + 0.67 + 0.33]^{-1}$

$= [2]^{-1}$

= 1 / 2

Probability that a customer has to wait for service

= p [w>0]

$= \dfrac{\phi^c \times P_0}{[c\,[1 - \phi / c]} = \dfrac{(0.67)^2 \times ½}{[2![1 – 0.67/2]}$

$= \dfrac{0.4489}{1.33\text{x}2} = \dfrac{0.4489}{2.66}$

= 0.1688

1.8 Check your progress:

There are some activities to check you progress. Answer the followings:

1. In a __________ one queue is served by one service facility.

a) single-channel, single-phase system
b) single-channel system
c) multichannel, single-phase system
d) Finite population
e) unlimited channel

2. The ___________ distribution is sometimes used to describe the time between arrivals.

a) Poissson
b) Exponential
c) Negative Exponential
d) Binomial
e) Chi Square

3. Arrivals are considered ________ when they are independent of one another and their occurrence cannot be predicted exactly.

a) Exponential 4.
b) r Unpredictable
c) Poisson
t
d) h Random
e
e) I uncontrollable
n

4. In the A/B/C designation for queuing systems, the B term represents information about __________.

a) population type
b) Arrival rate
c) Service time
d) Number of channels
e) Size of queue

5. When determining the arrival rate (λ) and the service rate (μ), the ________________ must be used

a) Model
b) Same time period
c) Little's flow equation
d) Kendall's notation
e) Simulation methodology

1.9 Summary

Queuing theory is the mathematical study of waiting lines, or queues. A Queuing model is constructed so that queue lengths and waiting time can be predicted. Queuing theory has its origins in research by Agner Krarup Erlang when he created models to describe the Copenhagen telephone exchange.

There may be different types of queue mainly as followings:
Structured queues
Unstructured queues
Kiosk based queues
Mobile Queue
Physical barrier
Signage and signaling systems
Automatic queue measurement systems
Information / customer arrival

A queuing system is specified completely by few basic characteristics. The foremost is the input process. It expresses the mode of arrival of customers at the service facility governed by some probability law. The number of customers emanate from finite or infinite sources broadly speaking, a queuing system occurs any time 'customers′ demand 'service′ from some facility; usually both the arrival of the customers and the service times are assumed to be random.
The ergodic conditions give the restrictions on the parameters under which the system will eventually reach the equilibrium.

The most common measures of system performance associated with queuing lines are the average number of customers waiting (in line of in a system), the average time customers wait (either in line or in the system), system utilization (the percentage of capacity utilized), the implicit cost of a given level of capacity

1.10 Keywords:

Queue**:** a queu**e** is a line of people or things waiting to be handled, usually in sequential order starting at the beginning or top of the line or sequence. In computer technology, a **queue** is a sequence of work objects that are waiting to be processed.

Queuing theory: is the mathematical study of the congestion and delays of waiting in line.

Customer: Refers to any thing that arrives at a facility and requires services

Server: refers to any resource that provides the requested services

1.11 Self Assessment Test:

1) The arrival rate of customers at a banking counter follows a poisson distribution with a mean of 45 per hors. The service rate of the counter clerk also poisson distribution with a mean of 60 per hours.

   (a) What is the probability of having Zero customer in the system (P_0).
   (b) What is the probability of having 5 customer in the system (P_5).
   (c) What is the probability of having 10 customer in the system (P_{10}).
   (d) Find Ls, Lq, Ws and Wq

2) Vehicles pass through a toll gate at a rate of 90 per hour. The average time to pass through the gate is 36 seconds. The arrival rate and service rate follow poisson distribution. There is a complaint the vehicles wait for long duration. The authorities are willing to install one more gate to reduce the average time to pass through the toll gate to 30 seconds if the idle time of the toll gate is less than 10% and the average queue length at the gate is more than 5 vehicles. Vehicle whether the installation of second gate is justified?

3) At a central ware house, vehicles arrive at the rate of 24 per hours and the arrival rate follows poisson distribution. The unloading time of the vehicles follows exponential distribution and the unloading rate is 18 vehicles per hour. There are 4 unloading crews. Find the following.

a) P_0 and P_3

b) L_q, L_s, W_q and W_s

4) Explain Queuing Discipline

5) Describe the Queuing models (M/M/1) : (GD/∞ /∞) and (M/M/C) : (GD/ ∞ / ∞)

6) Cars arrive at a drive-in restaurant with mean arrival rate of 30 cars per hors and the service rate of the cars is 22 per hors. The arrival rate and the service rate follow poisson distribution. The number parking space for cars is only 5. Find the standard results.

7) In a harbour, ship arrive with a mean rate of 24 per week. The harbour has 3 docks to handle unloading and loading of ships. The service rate of individual dock is 12 per week. The arrival rate and the service rate follow poisson distribution. At any point of time, the maximum No. of ships permitted in the harbour is 8. Find P_0, Lq, L_s, W_q, W_s

1.12 Answers to check your progress:

1. **(a)** single-channel, single-phase system
2. © Negative Exponential
3. (d) Random
4. © service Time
5. (b) Same time period

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Subject: Management Science	
Course Code: MBA 206	Author: Prof (Dr.) Hemant Sharma
Lesson No.: 9	Vetter:
Inventory Management	

Structure

9.1 Introduction

9.2 Types of inventory

9.3 Inventory decisions

9.4 Types of inventory costs

9.5 Inventory management systems

9.6 Economic manufacturing batch size

9.7 Safety stocks

9.8 Inventory model with purchase discount

9.9 Check your progress

9.10 Summary

9.11 Keywords

9.12 Self assessment Test

9.13 Answers to check your progress

9.14 References/ Suggested Readings

Learning Objectives:

After reading this chapter, the students will be able to understand the:

- Basic functions of inventory management;
- Different types of inventory;
- Various types of inventory costs;
- Different approaches to control the inventory level;
- The concept of safety stocks

9.1 Introduction:

Inventory is basically working capital and that is why control of inventories is very important as part of operations management. Inventories are crucial for proper functioning of manufacturing and retailing organizations. There are many types of inventories like raw material, spare parts or consumables, work-in-progress and finished goods. It is not necessary that every organization needs these resources but should work according to the needs and requirements of the resources depending upon what type of production is taking place.

Various departments within the same organization have a contradictory approach towards these kinds of resources. This is because the functions that are performed in various departments influence the motivation in them. For example, the sales department might need large amount of stocks of materials so that the production systems run very smoothly with any hesitation. On the other hand, the finance department would need a minimum investment in stocks so that the funds could be used elsewhere for even better purposes to enhance the performance of the organization.

There are different inventory systems that determine the when to order and how much to order. In this chapter, we will discuss all of them in detail.

9.2 Types of Inventory:

Inventories are used for many different purposes and by various departments for their respective needs and requirements, but there are generally five types of inventories that every production organization should emphasis on:

1) Movement inventories
2) Buffer inventories
3) Anticipation inventories
4) Decoupling inventories
5) Cycle inventories

1) Movement Inventories:

Everyday resources are being transported to the industries and putting them to use by production organization through various modes of transportation. Movement inventories are also called transit or pipeline inventories. This is basically dealt with transporting the resources from source to destination. For example, coal is transported from coalfields to an industrial township by trains, then the coal, while being transported will not be able to provide any service to the customers for power generation or for burning furnaces etc.

2) Buffer Inventories:

These inventories are basically kept for future needs for the organization in stock because there may a case when more inventories would be needed and therefore every organization keeps an average amount of inventories in stock so that the organization can utilize those resources efficiently and effectively without any delay. This mainly calls for uncertainty in demand, as every organization would need the required amount of stock but what would happen when the stock runs out? Everything would stop mainly the production so it is very important that excess amount of resources should be kept in stock. Similarly, the average time for delivery that is (the time between placing the order of resources and receiving those orders and getting them ready for use in stock, technically known as lead-time).

The idea of keeping buffer stocks is to enhance the level of providing customer service and gradually reducing the number of stock outs and back-orders. Stock out is something when the stock runs out and the needs of customers are not being able to fulfilled but in some situations back ordering is possible that is (the order for goods demanded is fulfilled as soon as the next shipment of stock arrives.) while in others it is not as it looks because the demand might be lost forever which leads to temporary or permanent loss of customer goodwill.
So it is very important to keep buffer stocks as demand may arise at any point of time.

3) Anticipation Inventories:

Anticipation inventories are put under scrutiny for future demands so that when the time arrives, the supply of products flow rapidly. Like producing rain coats before the rainy season, creating crackers before Diwali etc. The idea under this is to smoothen the flow of production process for longer time on an iterative scale instead of operating with excess overtime in a particular period and then keeping the system idle for long or even shut down the system because of unnecessary demand for another period.

4) Decoupling Inventories:

This type of inventory deals with the work rate of different machines and people because normally machines work at different rates- some slower and some faster. For example, a machine might be producing half the output of the machine on which the item being handled is to be processed the next. Inventories in between the various machines are held in order to disengage the processing on those machines. In absence of those inventories, different machines and people cannot work on a continuous basis. Clearly, therefore the decoupling inventories act as shock absorbers and have a cushioning effect in the face of varying work rates, and machine breakdowns and failures and so on.

5) Cycle Inventories:

Cycle inventories are those when purchases in lots instead in exact amount of stock need in a specific point of time. But yes if all purchases are made as per the exact requirement of stock there would have been no cycle inventories. But then the cost in getting these stocks would be much higher as per the customer needs and requirements. They are also called lot-size inventories and larger the lot-size inventory the greater would be the level of cycle inventory.

9.3 Inventory Decisions:

It is very important and is the top most priority of deciding about the inventories in a production organization as this would decide the future and present performance of the company. In any production organization deciding the inventories according the needs and requirements of it is very important. This can enhance the performance or bring down the efficiency.

So there are specific things any production manager should keep in mind before making decisions. They are:

- How much to order? – This is decided by the manager as to how much quantity to order for optimal performance and effective utilization of resources.
- When to order? – This is the most important aspect the manager should emphasize on because this would decide when should the products be ordered.

- ❖ How much stock should be kept in safety? – This indicates how much quantity should be taken under consideration so that the stock can be used safely in the future without any hesitation.

9.4 Types of Inventory Costs:

For deciding the best suitable inventory policy, the top most criteria used is the cost function. This inventory analysis has four major components:

1) Purchase Cost:

This is basically the nominal cost of an inventory. It is the cost incurred in buying from the outside sources, and it would be known as production cost if the items are produced within the organization. The cost is constant for a unit but may vary according to the quantity purchased increases or decreases. For example, the unit price is Rs.20 for up to 100 units and Rs.19.50 for more than 100 units. If a unit cost is constant, the control decisions would not have any affect because whether all the requirements are produced just once or made in installments the total amount of money involved would be the same.

2) Ordering Cost/Set-up Cost:

This occurs whenever the stock replenishes. It associates with the processing and chasing the purchased order, transportation, and inspection for quality. It is also called procurement cost. The parallel of ordering cost when the units are produced within the organization is the set-up cost. It refers to cost incurred in relation to developing production schedules. The ordering cost and set-up cost are taken to be independent to the order size. So the unit ordering/set-up cost decreases as the purchase order increases.

3) Carrying Cost:

Carrying cost is also known as holding cost and it refers to the cost that is associated with storing an item in the inventory. It is proportional to the amount of inventory and the time taken to hold that inventory. The elements of carrying cost include opportunity cost, obsolescence cost, deterioration cost. The carrying cost is expressed in terms of rate per unit or as a percentage of the inventory value.

4) Stockout Cost:

Stock out cost is the cost, which incurs when customers are not being served. These costs imply shortages. If stock out is internal, that means that some production is lost internally also resulting in idle time for man and machines. If stock out were external, it would result in potential sales or loss of customer goodwill. When the new shipment arrives, a customer who was denied earlier would be immediately supplied the goods. But it would involve costs like packaging costs and shipment costs.

9.5 Inventory Management Systems:

There are basically two types of management systems:

- ***Fixed order quantity system:*** Also known as re-order point, when a specific level is reached called the re-order level and the stock level reached this point, an order for a particular number of units is placed;
- ***Periodic Review System:*** This is a system where the stock is replenished over a fixed period of time. In this system, the time after which the order is placed, is fixed, but not the quantity.

Fixed Order Quantity System:

This system also called the Q-System. In this, a re-order point is established and as soon as the stock level reaches this level, new set of orders are placed. This system is taken under consideration of certainty. A couple of models based on different conditions shall be developed to study various operations of the system under deterministic conditions.

Model 1: The Classical EOQ Model

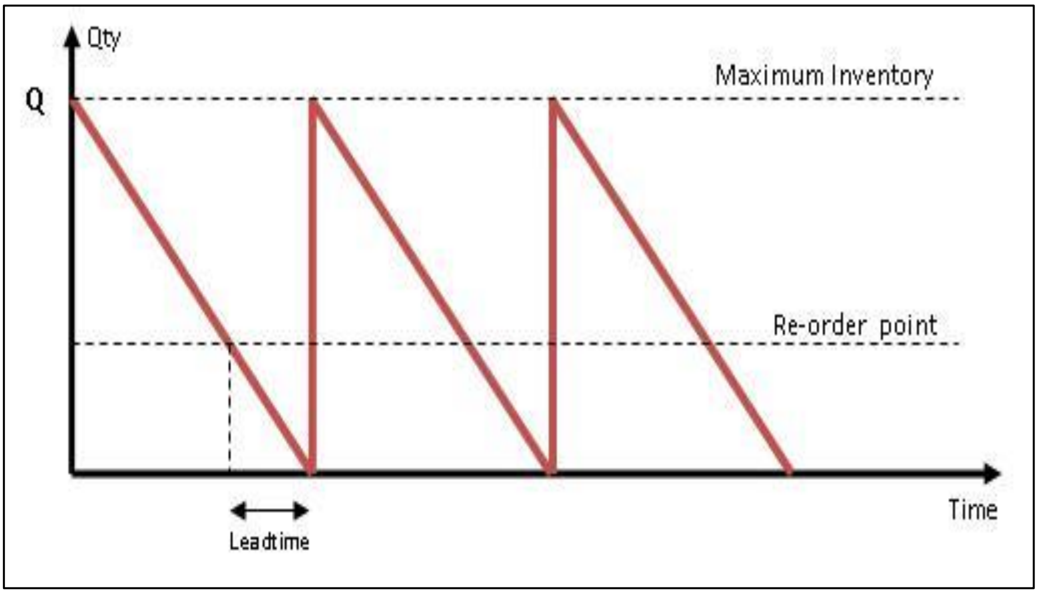
EOQ stands for Economic Order Quantity also known as the Wilson Formulation. It is the most elementary of all the inventory models. For this, a fixed cost model is made and then it is manipulated to form an inventory model.

This model is based on the following assumptions:

1) The demand for the item is continuous, constant and certain over time.
2) The purchase price is constant, and no discount is available on the large lots.

3) The inventory is replenished immediately as the stock level reaches level equal to zero. So there is no shortage or overage.
4) The lead is always known and fixed. When the lead-time is zero, the delivery of item is instantaneous.
5) Within the range of quantities ordered, per unit holding cost and the ordering cost are constant and thus independent of the quantity ordered.

With these assumptions, the inventory level would vary over time as shown in the graph 1 as below:



Graph 1

Now, we begin with a stock of Q on the time zero. This will be consumed at the rate of some units per day. If the stock can be replenished instantaneously (that means lead time is zero), then a new set of orders is made and the inventory is obtained. When this stock is consumed, an order would be made at another time.

The interval between two different points when orders are placed, or the time elapsed in consuming the entire lot of items, is called the inventory cycle. The maximum inventory held would be Q while the minimum be zero, and hence the average inventory level would be equal.

There is no need for maintaining a safety stock because of the first two assumptions. For determining the optimum order quantity, we shall take two types of cost: ordering cost and the holding cost. Since the purchase price is uniform in nature, it does not affect the decision as to the quantity of the item to be ordered for purchase and, hence, is irrelevant for the purpose.

The cost model assuming for a period of one year is:

T (Q) = O (Q) + H (Q)

Where,
Q = the ordering quantity
T (Q) = total annual inventory cost
O (Q) = total annual ordering cost
H (Q) = total annual holding cost

Example 1:

Samsung Electronics Co produces 2000 TV sets in a year for which it needs an equal number of picture tubes of a certain type. Each tube costs Rs10 and the cost to hold a tube in stock for a year is Rs 2.40. Besides, the cost of placing the order is Rs 150, which is not related to its size.

Now, if an order for 2000 tubes is placed, only one order per annum is required. When 1000 units are ordered, 2 orders in a year are needed, while 500 units are ordered to be supplied, then a total of 4 orders per annum are required. Naturally, as the number of orders placed increases the ordering cost goes up. More orders, however, would also imply smaller order quantity and therefore decreasing holding costs. Thus, we have a trade-off between the ordering and the holding cost. What we attempt in our EOQ model is, then, to find the order size that minimizes the cost function T (Q).

A) *Total Annual Ordering Cost:* This is given by the number of times an order is placed, N, multiplied by ordering cost per order. A.

O (Q) = N x A

The value of N itself is independent on the order quantity Q, and the annual demand, D. Here N would be equal to D/Q. Accordingly:

O (Q) = D/Q x A

So, When:

N=1, Q=2000 and O (Q) = 1x150 = Rs150

N=2, Q=1000 and O (Q) = 2x150 = Rs300

N=4, Q=500 and O (Q) = 4x150 = Rs600

N=5, Q=400 and O (Q) = 5x150 = Rs750

B) ***Total Annual Holding Cost:*** The annual holding cost is obtained by multiplying the unit holding cost, h, by the average number of units held in the inventory. As been pointed out earlier, the average inventory held equals Q/2. Consequently, the total cost of holding inventory, per annum would be:

H (Q) = Q/2 x h

So, When:

Q=2000, H (Q) = 2000/2 x 2.40 = Rs 2400

Q=1000, H (Q) = 1000/2 x 2.40 = Rs 1200

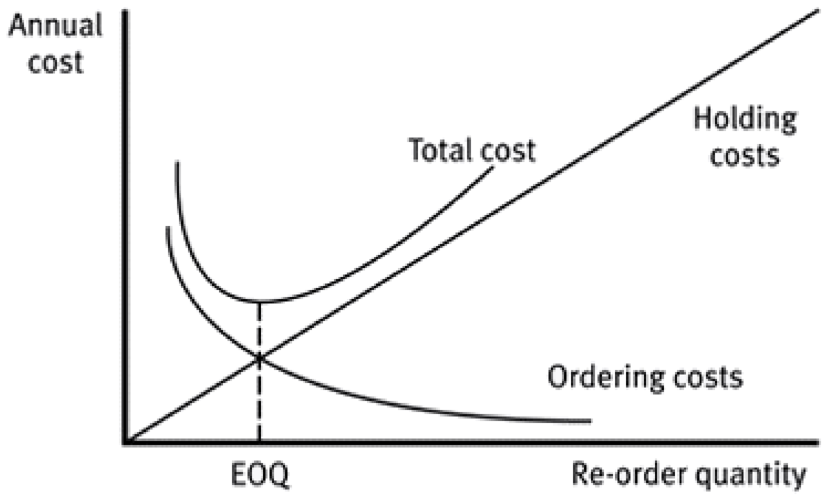
Q=500, H (Q) = 500/2 x 2.40 = Rs 600

Q=400, H (Q) = 400/2 x 2.40 = Rs 480

This may be pointed out that although the cost of holding a unit in an inventory is given in this question, more often the holding cost is expressed as a proportion, or percentage of the value of inventory. It may be stated for example: that the inventory holding costs are 15% per annum of the value of an item. It implies that if an item costs Rs 40, then the holding cost would be 15% of 40 = Rs 6 per unit per year.

C) ***Total Annual Inventory Cost:*** Both the cost components can now be added up and we can obtain the total cost of inventory.

The total cost curve is obtained by adding two components O (Q) and H (Q). The minimum point on this curve determines the optimal quantity, for which each order is placed each time. This ensures the minimization of total cost.



Graphic Determination Of EOQ

The graphic and tabulation methods of determining the EOQ are cumbersome.

We can obtain this value using the following formulae:

$Q = \sqrt{2AD}/h$

Or, $Q = \sqrt{2AD}/ic$

For example: We have A = Rs 150 per order, h = Rs 2.40 per unit per annum. D = 2000 units. Thus,

Q= $\sqrt{2} \times 150 \times 2000/2.40$

$=\sqrt{2{,}50{,}000} = 500\ units$

Determination of the Re-order Level:

The re-order level would be known at a point such as: The data below,

No. of working days = 250

Lead time = 15 working days

With this info, the daily demand = 2000/250 = 8 tubes

Demand during lead time = 15 x 8 = 120 tubes

Re-order level=120 tubes

A) Annual Total Variable Inventory Cost: The minimum annual inventory cost can be determined by substituting Q* for Q.

   T (Q*) = D/Q* x A + Q*/2 x h

   It may be noted that when the holding cost is expressed in the proportion form, we have: T (Q*) = $\sqrt{2}Adic$

B) Inventory Cycle: With a uniform and constant demand D, and the economic order quantity Q*, the problem of the optimal interval between the successive orders can be answered easily. If T* represents the optimal interval between any consecutive orders, we have,

   T*=Q*/D

   T* is also called the inventory cycle time.

C) Number of Orders: The optimal number of orders placed per year, N*, can also be obtained. It equals the reciprocal of T*. Thus, N* = 1/T*. Thus, when T* = 0.25 = 1/4, N* = 4 orders per year.

D) Rupee value: The monetary value of optimal order quantity and average inventory held can also be determined:

   Rupee value of EOQ = Q* x C (where c is the unit price)

   = 500 x 10 = 5000

   Rupee value of the average inventory = Q* x C/2

   = 500 x 10/2 = Rs 2500

In most cases, demand is expressed in money terms instead of units. So in this case, of the unit price is known, the demand may be converted into units by dividing rupee demand by the unit cost price.

Where ever, the cost is not given and then we can determine the economic order quantity in rupee terms. When the demand is given in monetary terms, the holding cost must be expressed as a proportion.

D_m= the annual demand in rupee terms

A= the acquisition cost

I = the holding rate

Example:

Using the following data, obtain the EOQ and the total variable cost associated with the policy if ordering quantities of that size.

Annual Demand = Rs. 20000

Ordering Cost = Rs 150 per order

Inventory carrying cost= 24% of average inventory value

Here,

D_m = Rs20000

A = Rs 150/order

I = 24% = 0.24

$$\text{EOQ (in rupees)} = \frac{\sqrt{2x150x20000}}{0.24}$$

$$= \text{Rs } 5000$$

$$\text{Total Cost, T (Q*)} = \sqrt{2} \times 150 \times 20000 \times 0.24$$

$$= \text{Rs } 1200$$

Violation of Assumptions of EOQ Model:

- In the EOQ model, we assumed that demand of an item is certain, continuous and constant. But however, the demand is more likely to be uncertain, discontinuous and variable.
- Demand is always supplied immediately and there is no availability of shortage. However, even when the demand and lead-time are known and constant, stockouts may be permitted.

- The unit price is the same. The analysis can be extended to cover situations when quantity discounts are available.
- The implicit assumption that the entire quantity ordered for would be received in a single lot may not hold true sometimes. If the supply of goods is gradual, the model needs adjustments.

<u>Model 2</u>: EOQ with Price Breaks

The previously discussed, the classical EOQ model is based on the assumption that the cost of an item under consideration is uniform. But in real life, it is very common to find cost discounts on quantities for which the order is placed. Lower rates are highlighted if the quantity of goods is high. So in cases like these, the quantity ordered should be carefully examined taking into consideration the price levels of different quantity ranges.

When the unit cost price is uniform, the purchasing cost is inadequate to determine the order size. But under the conditions of price break, the item cost, being a function of order quantity, is the incremental cost and must be included in the cost model. As such, the cost model would include the holding cost, ordering cost and the purchasing cost of items.

$$T(Q) = \frac{D}{Q} \times A + \frac{Q}{2} \times h + ciD$$

This cost model is a step function, and not a continuous like the one given earlier. To understand how optimal order quantity can be determined in such a case, we would take an example:

D = 2000 units per annum

A = Rs 150 per order

h = 2.40 per unit per annum

Suppose now that the supplier informs that if the order size is at least 800 units, he is prepared to supply it at a discounted rate of Rs 9.80 per tube.

With the EOQ = 500 units

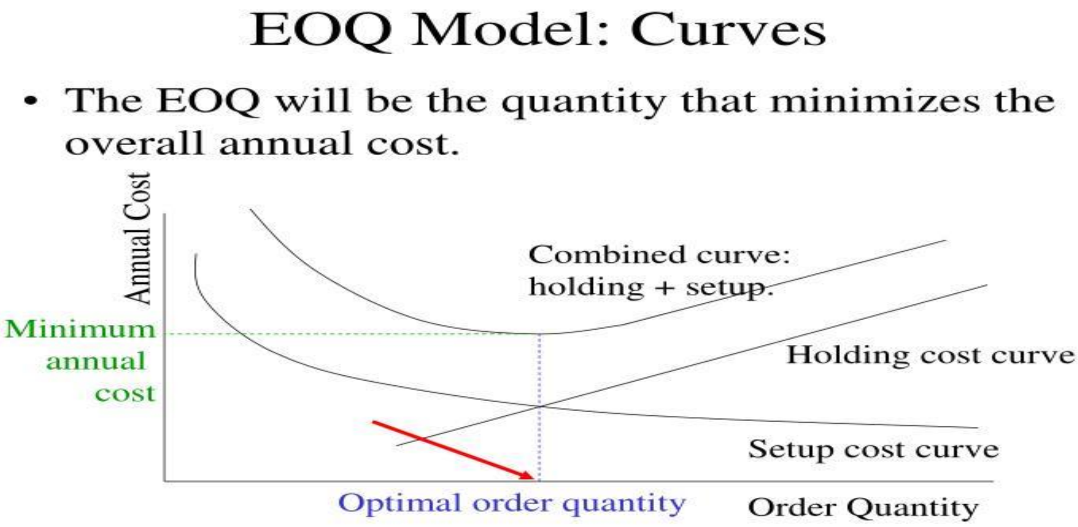
T (Q) = $\frac{2000}{500} \times 150 + \frac{500}{2} \times 2.40 + 10 \times 2000$

= Rs 20935

With the EOQ = 800 units

T (Q) = $\frac{2000}{800} \times 150 + \frac{800}{2} \times 2.40 + 9.80 \times 2000$

= 750/2 + 960 +   19600 = Rs 20935



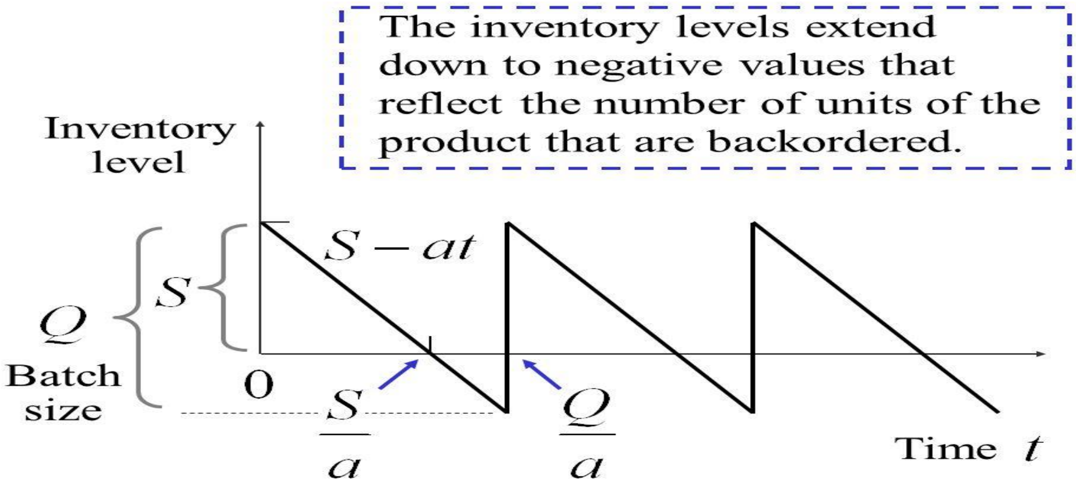
Cost Curve For the Price-Break Model

Clearly, the curve shows a sizeable drop in the cost due to the price discount at a quantity of 800 units. At this level, the total cost is lower than the total cost corresponding to 500 units.

Model 3: Inventory Model with Planned Shortages

In general inventory situations, a shortage is mainly undesirable and should be avoided because shortages can result in loss of customer goodwill, reduction in future orders, it may result in unfavorable changes in the market share etc. and in some situations, customers tend to move from a source to another for different requirements, and also customers may not withdraw the orders and wait until the next shipment arrives. This situation is also called the *back-ordering situation*. The EOQ model assumes that the inventory is replenished precisely when the inventory level falls off to zero. With the assumption of back ordering, Shortages, and therefore, the cost of shortage is not considered in that model. It may be advisable on economic considerations, specially, when the value of the item in question of setting off the cost of shortages against the saving in the holding cost.

Graph below shows negative inventory (zero level) i.e. number of units backordered. As soon as the lot of Q items is received, the customers whose orders are pending would be supplied their needs immediately and as such the maximum inventory level would be Q-S.



The EOQ Model with Planned Shortages

In developing the cost function, we would consider cost of shortages in addition to the holding and the ordering costs. Cost of shortages or the backordering cost is incurred in terms of the labour and special delivery expenses and the loss of customer goodwill.

Total Cost = Ordering Cost + Holding Cost + Shortage Cost

Ordering Cost: As seen before, if the cost of placing an order be A, and the total demand be D, we have,

Annual ordering Cost = D/Q x A

Holding Cost: It is the period in a given inventory cycle when positive inventory is held. Since the maximum inventory, M, is Q-S, the average inventory level equals (Q-S)/2. Thus,

Holding cost during a given cycle T = (Q-S)/ 2 x ht

From the above formulae, we observe that the quantity (Q-S) is sufficient to last a period.

Shortage Cost: We shall now develop expressions for the average number of shortages and the shortage cost with the help of which we shall determine the annual shortage cost. Since S represents the maximum level of shortages, the average level of shortages, during the period when there is a shortage shall be S/2.

From the analysis, the total cost expression would be:

T (Q) = $\frac{D}{Q}A + (Q - S)^2 h/2Q + bS^2/2Q$

Derived from the expression, Q* would be:

Q* = $\frac{\sqrt{2AD}}{h} \times \left(h + \frac{b}{b}\right)$

= $\sqrt{2AhD} + \sqrt{\frac{B}{H} + B}$

Determination of the re-order level: The optimal shortage being 474 units and the consumption during the lead-time being equal to 8x15=120 units, the re-order level would be established at a point where the shortage reaches 474-120=354 units.

Therefore: Re-order level = -354 units (shortage level of 354 units).

9.6 Economic Manufacturing Batch Size:

The EOQ concept can be further extended to the determination of optimal manufacturing batch size for semi-finished and finished goods. If the batch size is large, then the average level of inventory is also large therefore the carrying costs for the inventory are high. But a few cases like, large batches of would suffice for the annual requirements, the number of set-ups would be low. On the other hand, when batch size is small, the order cost is higher, but at the same time, the average inventory level is smaller thus making the carrying cost lower. Thus, there is clear trade-off between costs involved.

Cost of Setup:

The set-up cost mainly includes the following:

1) Cost of time spent in setting up the equipments and organizing the labour for a manufacturing batch. This is the cost of the idle time of labour and the machinery, which would have otherwise produced goods. This is the opportunity cost of the time lost due to a set-up.
2) Cost due to rejects, scrap, rework generated during a set-up.
3) Variable cost of administrative paper work for a set-up.

Calculation of Economic Batch Quantity:

The Economic Batch Quantity (EBQ) Formula for a single product is:

This expression is similar to that derived for the classical inventory model except for the fact that it takes into consideration production and consumption rates of the product.

Example:

Compute the EBQ for manufacture given the following data:

Monthly demand = 500 units

Daily production rate = 25 units

Days in a month = 25 days

Cost of set–up = Rs. 1,500

Cost of holding inventory = Rs.10 per unit per year

Solution:

Annual Demand A = 500 x 12 = 6000 units per year

The daily consumption rate r = $\frac{\text{Monthly Consumption}}{\text{No.of days in a month}} = \frac{500}{25}$

=20 units per days

The above problem of optimal manufacturing batch size is confined to a case where only one product is being manufactured. In practice, a number of different products may be manufactured on the same plant facility. One might argue that the formula for the single product can be used to determine individually the optimal batch quantities for different products. Although this individual determination of the manufacturing batch sizes would produce most economical results as far as individual products are concerned, it might present some difficulties in a few cases.

When multiple products share the same plant facility, there are chances of interferences and therefore, stock-outs occur. Such interference between different products is experienced sometimes, when the products share the same equipment but the batch quantities are calculated independently. To avoid this kind of problem, it is suggested that the economic batch size of the products using the same plant facility be determined jointly. Therefore, there will be joint cycles of manufacture and in each joint cycle all the products will be manufactured in appropriate quantities. The determination of the economic batch sizes of the different products then amounts to the determination of the optimal number of joint cycles in a year; annual demand for a product divided by the optimal number of joint cycles gives the economic batch quantity for the product.

Since all the products, using the same plant facility, are manufactured, one after another in each optimal joint cycle, there is no question of shortage of any product at any time. The principle to be followed in joint cycle determination is similar to that for the determination of the optimal batch quantity for individual products. Numerical based on joint cycles are beyond the scope of this chapter.

9.7 Safety Stock

The inventory models discussed so far are based on the common assumption of constant and know demand for the item and the lead-time. Therefore, these models are called deterministic models. The models that consider the situation in which the demand and demand and/or lead-time are not known with certainty and they need not be constant is beyond the scope of this chapter. In these models, demand and lead time are taken as random variables, capable of assuming varying values whose probability distribution may be known.

In the models, the stock is replenished as soon as the stock reaches the point of exhaustion, due to the assumption underlying them. Under such idealistic situation, there is no need to maintain any extra stock because the supplies would reach the moments the stock level reduce to zero and there would be no stock outs (unless they are intentionally allowed to occur). However when the demand is varying and so is the lead time, there is a need to provide for the safety or buffer stock in order to meet either or both the lead time, there is a need to provide for the safety or buffer stocks in order to meet either or both the contingencies, viz. that demand rate during the lead time is in excess of what was expected/forecasted and that the delivery of good is delayed. The safety stock, then acts as a cushion against stock-outs caused by random deviations of nature.

The safety stock is an important constituent of the re-order level that is determined as the expected demand of the item during lead time plus the safety stock. If the demand varies about the mean daily demand equal to d with the expected lead time equal to L days, and we set the re-order level R at L units, then we should expected a shortage to occur in about half the lead time periods. To reduces this 50% probability of being out of stock, the safety stock SS would be required to be kept. Thus,

Re-order level, $R = L^* d + S.S.$

We know that in this system, an order is placed as soon as it reaches the re-order level. Therefore, how high or low is the rate of demand before the re-order level reaches is of little consequence. What is significant is the level of demand during the lead-time. Here fresh supplies are received as soon as the stock level reaches the safety level. In this kind of a situation, the average stock held would be exactly equal to $SS + Q/2$.

The idea of keeping the safety stock is clearly to prevent stock out and it is the amount of stock that the organization would always like preserve for meeting extraordinary situation. In general higher safety stock would be called for in situation where costs of stock out are larger; higher levels of service (i.e. meeting greater proportion of demand) are sought; significant variation are observed in the lead time and/or time demand; and where holdings costs are smaller. Naturally, the higher the level of safety stock the greater the service level and therefore to strike a balance between the two, The optimal safety stock level is determined where successively declining stock out costs and successively rising holding costs, caused by the successive units added to the safety stock, would balance.

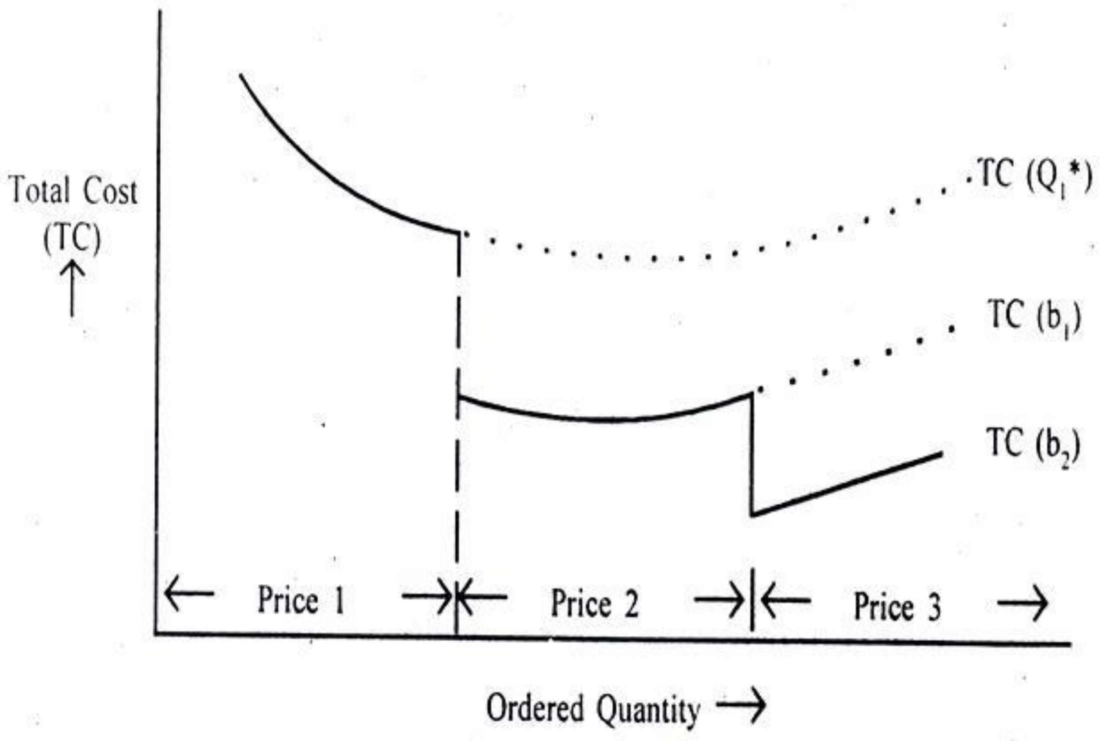
There is no rigid formulation for determining the optimum level of safety. The different approaches available for the purpose are based on the demand, the lead time and the stock out costs. The complexity of the situation is determined by the extent and nature is determined by the extent and nature of information available about these factors.

9.8 Inventory Model with Purchase Discounts:

The classical model for inventory does not take into account, amongst other things, the quantity discounts given by the supplier if material is purchased in bulk. As the discount might be relevant in the inventory analysis, this could be included in the total relevant cost and therefore in this case, the total cost function becomes:

Where is the supply price per unit of the inventory item and f is the carrying cost of the inventory expressed as a fraction of the inventory value.(Other nomenclature remains the same.)

Differentiating the total costs with respect to Q and equating the result to zero, we get the optimal procurement quantity.



At the two different prices (1 and 2) the $Q_{optimal}$ values are different. Which one shall we choose $Q_{optimal}$ for price 1 or $Q_{optimal}$ for price 2 the answer is not straightforward. We shall have to plot the total cost (relevant) with respect to the procurement lost size. For lot sizes less than b price 1 is operative and we get a total cost curve corresponding to it. For lot sizes equal to or greater than b we get another total cost curve. These curves need not exhibit minima within their zone. Price 1 total cost curve can have a minimum in the zone where price 2 is operating. Conversely. Price 2 total cost curve can have a minimum in zone where price 1 is operating.

It should be noted that due to the earlier given equations, the total cost curve for the second price will always be lower than the total cost curve for the first price, the minimum total cost for the second price will be lower than the minimum total cost for price 1, and the $Q_{optimal}$ for price 2 will always be higher than $Q_{optimal}$ for price 1. In spite of this, the three possibilities arise.

Here again, it is obvious that we choose $Q_{optimal}$ in fact $Q_{optimal}$ does not exit. The lowest of the total cost at price 1 is at price 1 is lot size 'b' and this total cost will have to be higher than the total cost for $Q_{optimal.}$

Here price 2 curve shows a minimum in the price 1 zone and the minimum is therefore imaginary. Hence, the only choice is between $Q_{optimal}$ and the price break quantity (at which the real part of the price 2 curve begins). This can be decided by comparing the total costs corresponding to the two choices.

The determination of the optimal quantity in the case of purchase discounts, therefore, follows the procedure given below:

1. Calculate $Q_{optimal}$ the optimal lost size corresponding to price 2.
2. Find out if the $Q_{optimal}$ falls in its own range. If so, desired optimal order quantity is $Q_{optimal}$ if not, carry out the following procedure.
3. Compare the total cost at $Q_{optimal}$ with the total cost corresponding to the lost size 'b' (price break quantity) at the second price. If the former is less than the latter,

Choose $Q_{optimal}$ Otherwise, the optimal order quantity is equal to the price-break point.

Example: The supply of a special component has the following price schedule.

0 to 99 item: Rs 1000 per unit

100 items and above: Rs 950 per unit

The inventory holding costs are estimated to be 25% of the value of the inventory. The procurement ordering costs are estimated to be Rs. 2,000 per order. If the annual requirement of the special component is 300, compute the economic order for the procurement of these items.

Steps 2 and 3:

Therefore, we have to determine the optimal total cost for the first price and total cost at the price-break point corresponding to the second price, and compare the two.

The total cost (optimal for the first price)

$$=\sqrt{2x2000x1000x0.25x300 + 1000x300}$$

=17,320+300,000=Rs.3,17,320

The total cost for the price-break point (corresponding to the second price):

$$TC=200 \text{ x } \frac{300}{100} + \frac{100}{2} \text{ x } 950 \text{ x } 0.25 + 950 \text{ x } 300$$

=6,000 + 11,875 + 285,000

=3,02,875

This is lower than the total cost corresponding to $Q_{\text{optimal.}}$

Therefore, the economic quantity for a procurement lot is 100 units (price-break point).

Consideration of Uncertainties:

In the above given models for the determination of 'normal' inventory consumption rates were assumed to be constant. In actual practice, there are always uncertainties stemming from two basic reasons:

1. Variability in sales, hence variability in the demand for the materials or the consumption of the materials
2. Delay in the supplies of raw materials.

9.9 Check your progress

1. The two most basic inventory questions answered by the typical inventory model are
   a. timing and cost of orders
   b. quantity and cost of orders
   c. timing and quantity of orders
   d. ordering cost and carrying cost

2. Which of the following statements about the basic EOQ model is true?

a. If the ordering cost were to double, the EOQ would rise.

b. If annual demand were to double, the EOQ would increase.

d. If annual demand were to double, the number of orders per year would decrease.

e. All of the above statements are true.

3. An inventory decision rule states that "when the inventory level goes down to 14 gearboxes, 100ngearboxes will be ordered." Which of the following statements is true?

   a. 100 is the reorder point, and 14 is the order quantity.

   b. The number 100 is a function of demand during lead time.

   c. 14 is the safety stock, and 100 is the reorder point.

   d. 14 is the reorder point, and 100 is the order quantity.

4. Which of the following would not generally be a motive for a firm to hold inventories? To

   a. take advantage of quantity discounts

   b. minimize holding costs

   c. decouple production from distribution

   e. meet anticipated demand

5. The annual demand for a product has been projected to be 2,000 units. This demand is assumed to be constant throughout the year. The ordering cost is $20 per order, and the holding cost is 20 percent of the purchase cost. The purchase cost is $40 per unit. There are 250 working days per year. Whenever an order is placed, it is known that the entire order will arrive on a truck in six days. Currently, the company is ordering 500 units each time an order is placed. What level of safety stock would give a reorder point of 60 units?

a. 10
b. 14
c. 18
d. None of above

6. The purpose of safety stock is to:
    a. adapt to uncertain demand
    b. adapt to uncertain lead time
    c. adapt to uncertain demand during lead time
    d. none of the above

9.10 Summary:

- Inventory serves a useful purpose in the manufacturing organizations. Firms can help minimize the need for inventory by carefully managing those factors that drive inventory levels up
- Inventory items can be divided into two main types: Independent demand and dependent demand items. The systems for managing these two types of demand, inventories differ significantly
- The two classic systems for managing independent demand inventory are periodic review and perpetual review systems
- The economic order quantity (EOQ) is the order quantity that minimizes total holding and ordering costs for the year. Even if all the assumptions don't hold exactly, the EOQ gives us a good indication of whether or not current order quantities are reasonable
- The reorder point formula allows us to determine the safety stock (SS) needed to achieve a certain cycle service level. In general, the longer the lead times are, and the greater the variability of demand and lead times, the more SS we will need
- Inventories are vital to the successful functioning of manufacturing and retailing organizations
- The basic questions to keep in mind before getting any inventory:
    a) How much inventory to keep

b) When to keep the inventory in the warehouse

- ❖ Buffer stock is kept for review period + lead-time
- ❖ Maximum inventory on hand is (Normal consumption + Buffer Stock) both for review period plus on order.

9.11 Keywords:

- ▪ ***Inventory:*** is working capital and therefore the control of inventories is an important aspect of operations management
- ▪ ***Lead Time:*** The time elapsing between placing an order and having goods in stock
- ▪ ***Procurement costs:*** associated with processing and chasing of an order, transportation, inspection for quality, expediting overdue orders and so on

9.12 Self-assessment test:

1. Define inventory. Discuss various types of inventory costs.
2. Discuss various types of inventories.
3. What are the basic assumptions underlying the classical EOQ model? Also discuss its limitations.
4. What is the set-up cost of manufacture?
5. Discuss economic batch quantity with suitable example.

9.13 Answers to check your progress:

1. a) timing and cost of orders
2. e) All of the above statements are true.
3. d) 14 is the reorder point, and 100 is the order quantity
4. b) minimize holding costs
5. b) minimize holding costs
6. c) adapt to uncertain demand during lead time

9.14 References/ Suggested Readings:

R.G. Schroeder, Operation Management- Decision Making in the Operations Function, McGraw-Hill, International Student Edu, 1985

J.K Sharma, Operations Management – Text and Cases, Trinity Press, 5th Edition

Ajay K Garg, Production and Operations Management, McGraw Hill

S.N. Charry, Production and Operations Management, McGraw Hill, 5th Edition

Kanishka Bedi, Production and Operations Management, Oxford, 2nd Edition

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Subject: MANAGEMENT SCIENCE	
Course Code: MBA 206	Author: Prof (Dr) Hemant Sharma
Lesson No.: 10	Vetter:
Network Design, Critical Path Method and PERT	

Structure

1.1 Introduction to Project Network and its Key Concepts

1.2 Construction of Project Network Diagramme

1.3 Critical Path Method (CPM)

1.4 Critical Path and Critical Activities

1.5 Programme Evaluation and Review Technique (PERT)

1.6 Measure of Certainty

1.7 Determination of Project Completion Time in PERT

1.8 Check your progress

1.9 Summary

1.10 Key Words

1.11 Self Assessment Test

1.12 Answers to check your progress

1.13 References- Suggested Readings

Learning Objectives

After Studying this lesson, students will be able to:

- Understand the key concepts of network diagramme
- Develop project network diagramme
- Understand the concept of critical path and critical activities
- Determine the project completion time
- Understand the importance of Programme Evaluation and Review Techniques (PERT)
- Find out the probability of completion of a project before a stipulated time

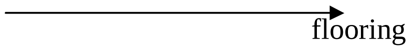
1.1 Introduction to Project Network and its Key Concepts:

Project network depicts various activities of any project in a particular sequence in which these activities are to be performed.

Certain key concepts pertaining to a project network are described below:

1. Activity

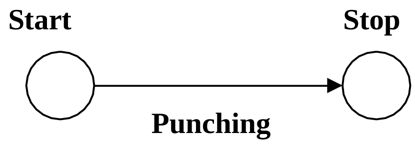
An activity means a work. A project consists of several activities. An activity takes time. It is represented by an arrow in a diagram of the network. For example, an activity in house construction can be flooring. This is represented as follows:



Construction of a house involves various activities. Flooring is an activity in this project. We can say that a project is completed only when all the activities in the project are completed.

2. Event

It is the beginning or the end of an activity. Events are represented by circles in a project network diagram. The events in a network are called the **nodes**. **Example:**



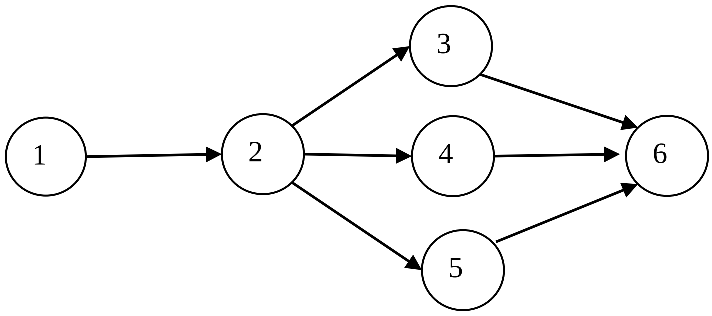
Starting a punching machine is an activity. Stopping the punching machine is another activity.

3. Predecessor Event

The event just before another event is called the predecessor event.

4. Successor Event

The event just following another event is called the successor event. **Example:** Consider the following.



In this diagram, event 1 is predecessor for the event 2. Event 2 is successor to event 1.
Event 2 is predecessor for the events 3, 4 and 5. Event 4 is predecessor for the event 6.
Event 6 is successor to events 3, 4 and 5.

5. Network

A network is a series of related activities and events which result in an end product or service. The activities shall follow a prescribed sequence. For example, while constructing a house, laying the foundation should take place before the construction of walls. Fitting water tapes will be done towards the completion of the construction. Such a sequence cannot be altered.

6. Dummy Activity

A dummy activity is an activity which does not consume any time. Sometimes, it may be necessary to introduce a dummy activity in order to provide connectivity to a network or for the preservation of the logical sequence of the nodes and edges.

7. Construction of a Project Network

A project network consists of a finite number of events and activities, by adhering to a certain specified sequence. There shall be a **start event** and an **end event (or stop event)**. All the other events shall be between the start and the end events. The activities shall be marked by directed arrows. An activity takes the project from one event to another event.
An event takes place at a point of time whereas an activity takes place from one point of time to another point of time.

1.2 Construction of Project Network Diagramme

Problem 1:

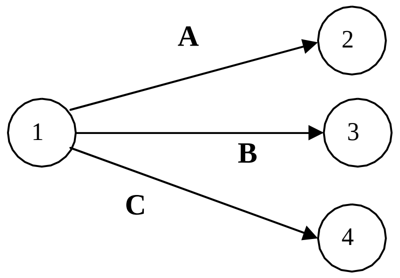
Construct the network diagram for a project with the following activities**:**

Activity	Name of activity	Predecessor Activity
1→2	A	-
1→3	B	-
1→4	C	-
2→5	D	A
3→6	E	B
4→6	F	C
5→6	G	D

Solution:

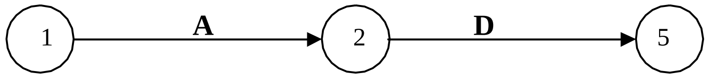
The start event is node 1.

The activities A, B, C start from node 1 and none of them has a predecessor activity. A joins nodes1 and 2; B joins nodes 1 and 3; C joins nodes 1 and 4. So we get the following:

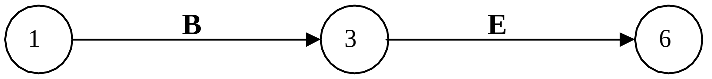


This is a part of the network diagram that is being constructed.

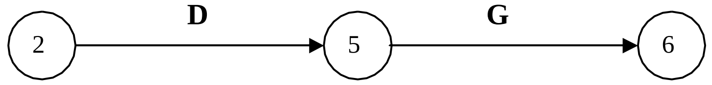
Next, activity D has A as the predecessor activity. D joins nodes 2 and 5. So we get



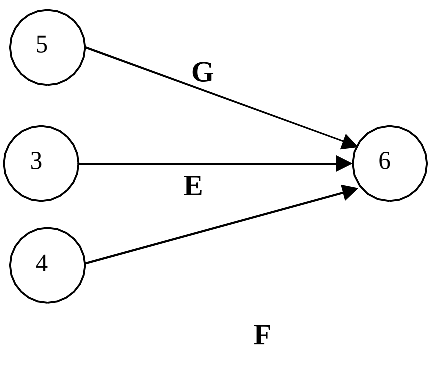
Next, activity E has B as the predecessor activity. E joins nodes 3 and 6. So we get



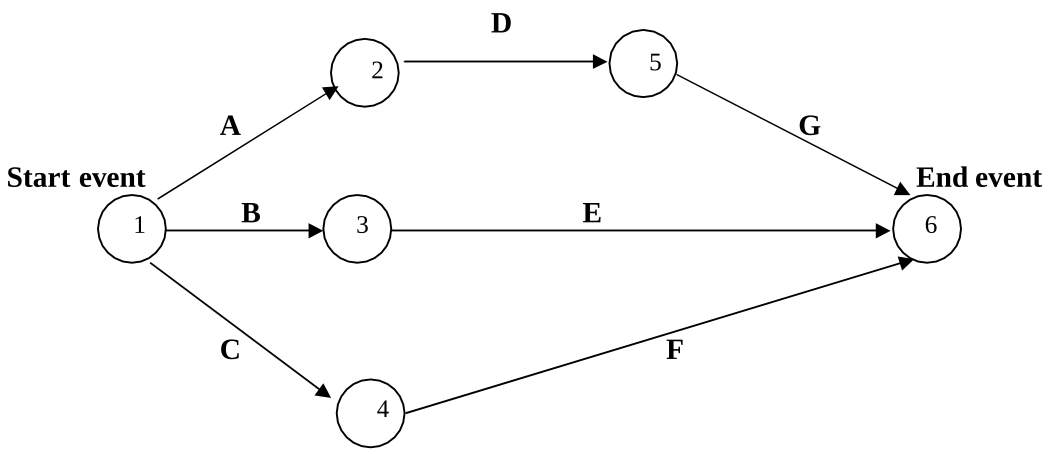
Next, activity G has D as the predecessor activity. G joins nodes 5 and 6. Thus we obtain



Since activities E, F, G terminate in node 6, we get



6 is the end event. Combining all the pieces together, the following network diagram is obtained for the given project:



We validate the diagram by checking with the given data.

Problem 2:

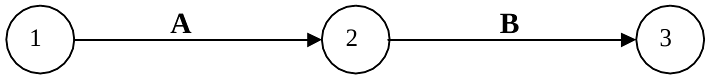
Develop a network diagram for the project specified below:

Activity	Immediate Predecessor Activity
A	-
B	A
C, D	B
E	C
F	D
G	E, F

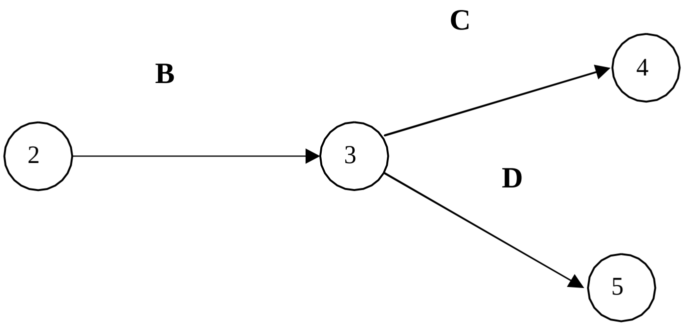
Solution:

Activity A has no predecessor activity. i.e., It is the first activity. Let us suppose that activity A takes the project from event 1 to event 2. Then we have the following representation for A:

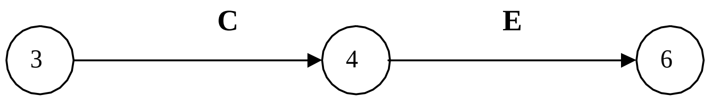
For activity B, the predecessor activity is A. Let us suppose that B joins nodes 2 and 3. Thus we get



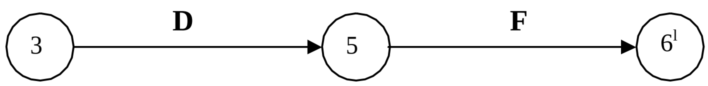
Activities C and D have B as the predecessor activity. Therefore we obtain the following:



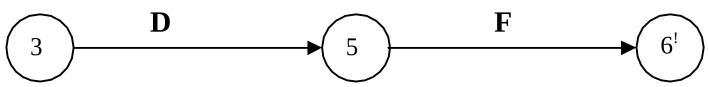
Activity E has D as the predecessor activity. So we get



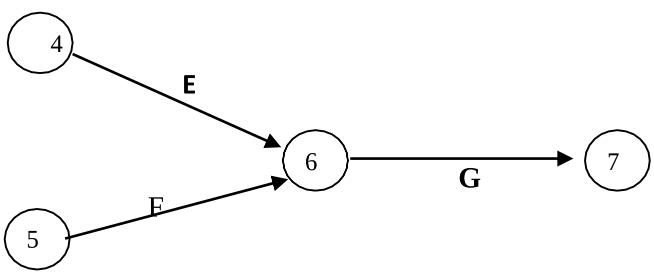
Activity F has D as the predecessor activity. So we get



Activity G has E and F as predecessor activities. This is possible only if nodes 6 and 6^{l} are one and the same. So, rename node 6^{l} as node 6. Then we get

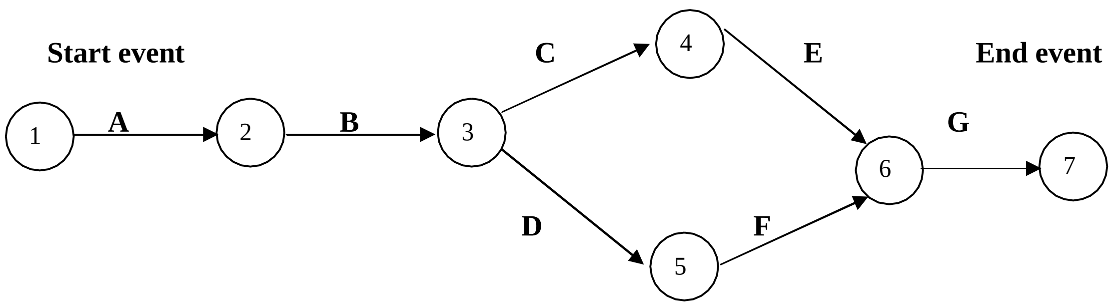


And



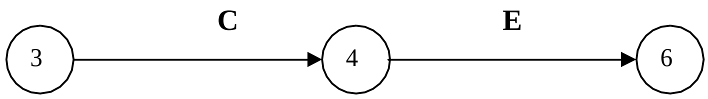
G is the last activity.

Putting all the pieces together, we obtain the following diagram the project network:

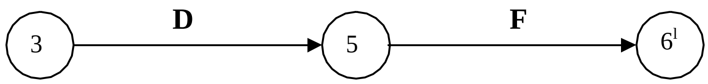


The diagram is validated by referring to the given data.

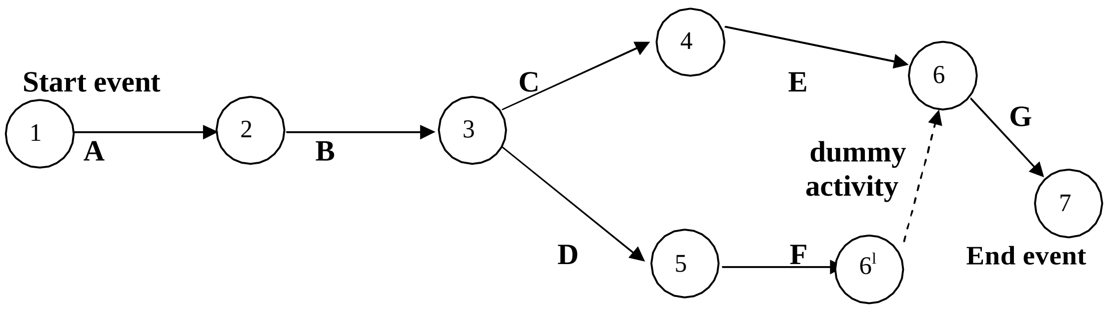
Note: *An important point may be observed for the above diagram. Consider the following parts in the diagram*



And



We took nodes 6 and 6^{l} as one and the same. Instead, we can retain them as different nodes. Then, in order to provide connectivity to the network, we join nodes 6^{l} and 6 by a dummy activity. Then we arrive at the following diagram for the project network:



1.3 Critical Path Method (CPM)

The critical path method (CPM) aims at the determination of the time to complete a project and the important activities on which a manager shall focus attention.

Assumption for CPM

In CPM, it is assumed that precise time estimate is available for each activity.

PROJECT COMPLETION TIME

From the start event to the end event, the time required to complete all the activities of the project in the specified sequence is known as the project completion time.

PATH IN A PROJECT

A continuous sequence, consisting of nodes and activities alternatively, beginning with the start event and stopping at the end event of a network is called a path in the network.

1.4 Critical Path and Critical Activities

Consider all the paths in a project, beginning with the start event and stopping at the end event. For each path, calculate the time of execution, by adding the time for the individual activities in that path.

The path with the largest time is called the **critical path** and the activities along this path are called the **critical activities** or **bottleneck activities**. The activities are called critical because they cannot be delayed. However, a non-critical activity may be delayed to a certain extent. Any delay in a critical activity will delay the completion of the whole project. However, a certain permissible delay in a non –critical activity will not delay the completion of the whole project. It shall be noted that delay in a non-critical activity beyond a limit would certainly delay the completion the whole project. Sometimes, there may be several critical paths for a project. A project manager shall pay special attention to critical activities.

Problem 1:

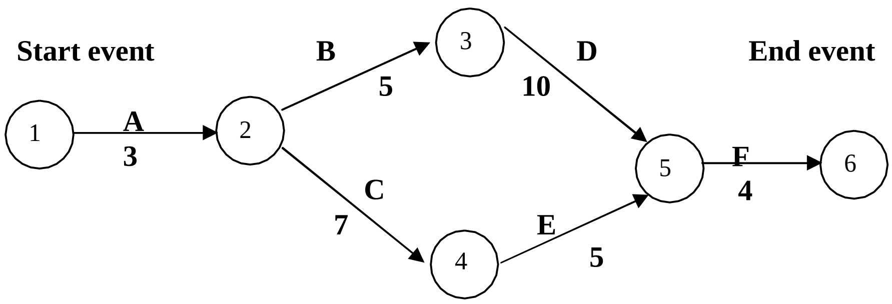
The following details are available regarding a project:

Activity	Predecessor Activity	Duration (Weeks)
A	-	3
B	A	5
C	A	7
D	B	10
E	C	5
F	D,E	4

Determine the critical path, the critical activities and the project completion time.

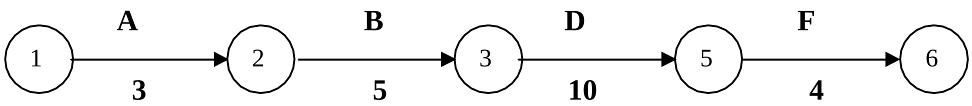
Solution:

First let us construct the network diagram for the given project. We mark the time estimates along the arrows representing the activities. We obtain the following diagram:



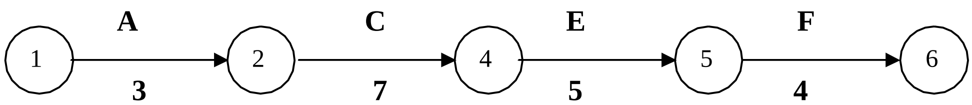
Consider the paths, beginning with the start node and stopping with the end node. There are two such paths for the given project. They are as follows:

Path I



With a time of 3 + 5 + 10 + 4 = 22 weeks

Path II



With a time of 3 + 7 + 5 + 4 = 19 weeks

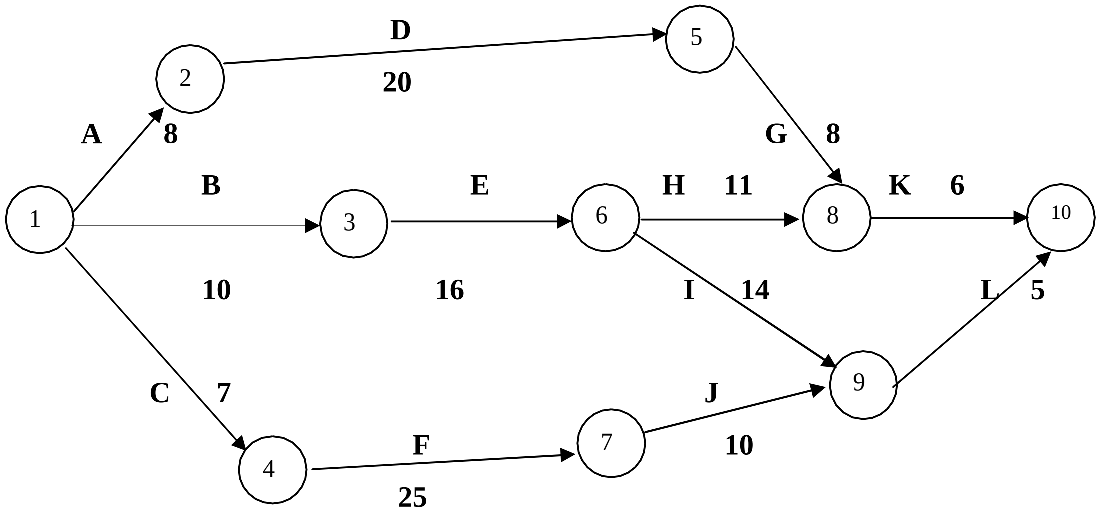
Compare the times for the two paths. Maximum of {22,19} = 22. We see that path I has the maximum time of 22 weeks. Therefore, path I is the critical path. The critical activities are A, B, D and F. The project completion time is 22 weeks.

We notice that C and E are non- critical activities. Time for path I - Time for path II = 22- 19 = 3 weeks.

Therefore, together the non- critical activities can be delayed upto a maximum of 3 weeks, without delaying the completion of the whole project.

Problem 2:

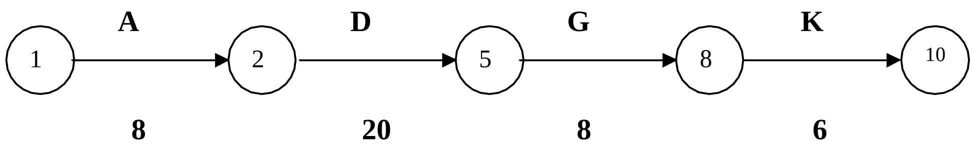
Find out the completion time and the critical activities for the following project:



Solution:

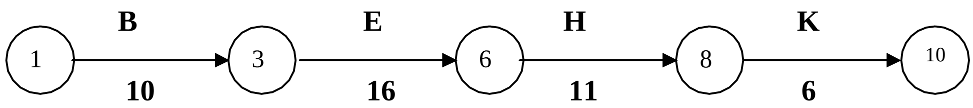
In all, we identify 4 paths, beginning with the start node of 1 and terminating at the end node of 10. They are as follows:

Path I



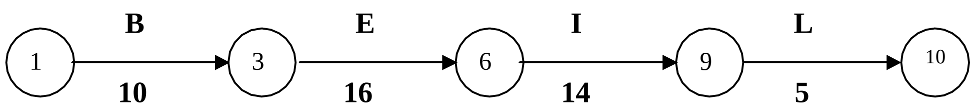
Time for the path = 8 + 20 + 8 + 6 = 42 units of time.

Path II



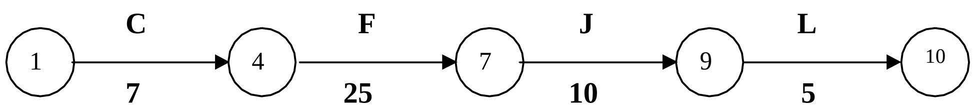
Time for the path = 10 + 16 + 11 + 6 = 43 units of time.

Path III



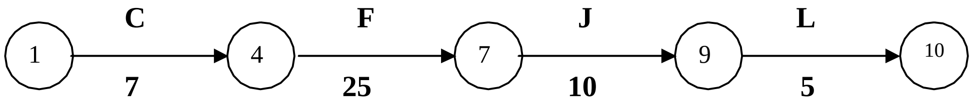
Time for the path = 10 + 16 + 14 + 5 = 45 units of time.

Path IV



Time for the path = 7 + 25 + 10 + 5 = 47 units of time.

Compare the times for the four paths. Maximum of {42, 43, 45, 47} = 47. We see that the following path has the maximum time and so it is the critical path:



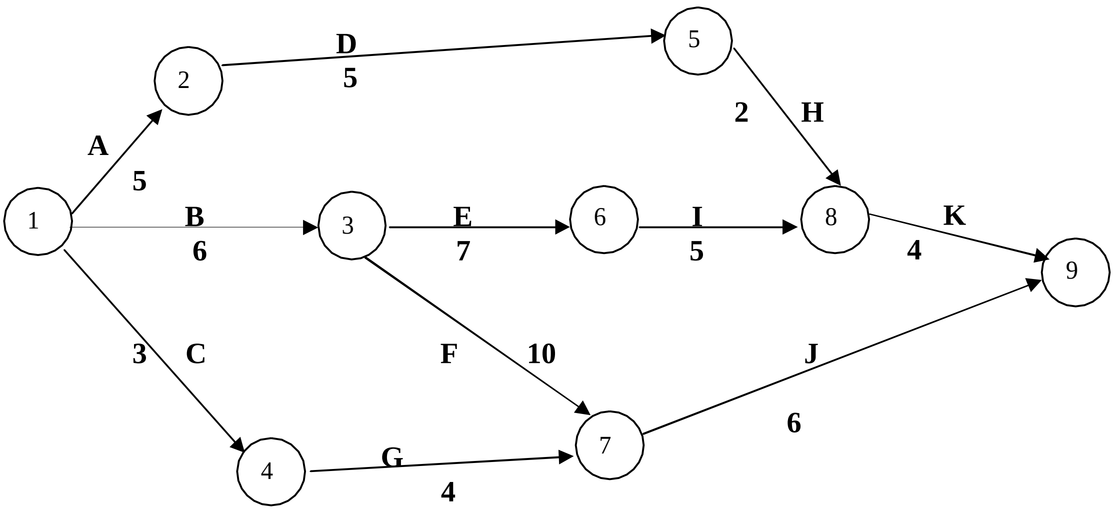
The critical activities are C, F, J and L. The non-critical activities are A, B, D, E, G, H, I and K. The project completion time is 47 units of time.

Problem 3:

Draw the network diagram and determine the critical path for the following project:

Activity	Time estimate (Weeks)
1- 2	5
1- 3	6
1- 4	3
2 -5	5
3 -6	7
3 -7	10
4 -7	4
5 -8	2
6 -8	5
7 -9	6
8 -9	4

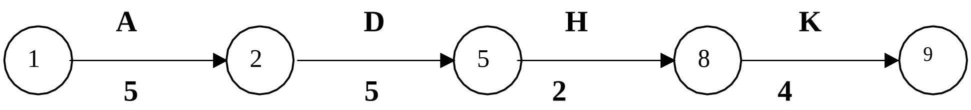
Solution: We have the following network diagram for the project:



Solution:

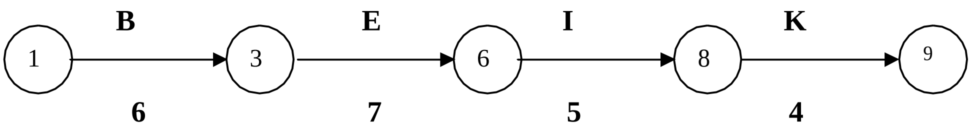
We assert that there are 4 paths, beginning with the start node of 1 and terminating at the end node of 9. They are as follows:

Path I



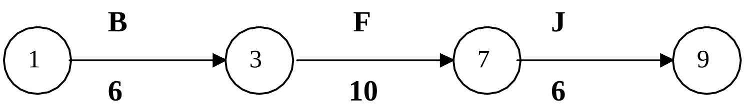
Time for the path = 5 + 5 + 2 + 4 = 16 weeks.

Path II



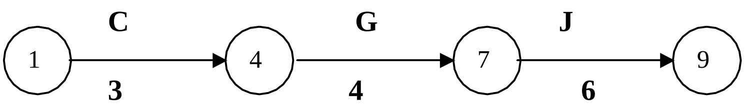
Time for the path = 6 + 7 + 5 + 4 = 22 weeks.

Path III



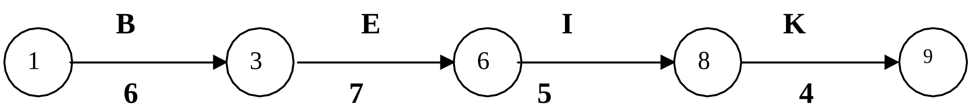
Time for the path = 6 + 10 + 6 = 16 weeks.

Path IV



Time for the path = 3 + 4 + 6 = 13 weeks.

Compare the times for the four paths. Maximum of {16, 22, 16, 13} = 22. We see that the following path has the maximum time and so it is the critical path:



The critical activities are B, E, I and K. The non-critical activities are A, C, D, F, G, H and J. The project completion time is 22 weeks.

1.5 Programme Evaluation and Review Technique (PERT)

Programme Evaluation and Review Technique (PERT) is a tool that would help a project manager in project planning and control. It would enable him in continuously monitoring a project and taking corrective measures wherever necessary. This technique involves statistical methods.

ASSUMPTIONS FOR PERT

Note that in CPM, the assumption is that precise time estimate is available for each activity in a project. However, one finds most of the times that this is not practically possible.

In PERT, we assume that it is not possible to have precise time estimate for each activity and instead, probabilistic estimates of time alone are possible. A multiple time estimate approach is followed here. In probabilistic time estimate, the following 3 types of estimate are possible:

1. Pessimistic time estimate (t_p)
2. Optimistic time estimate (t_o)
3. Most likely time estimate (t_m)

The optimistic estimate of time is based on the assumption that an activity will not involve any difficulty during execution and it can be completed within a short period. On the other hand, a pessimistic estimate is made on the assumption that there would be unexpected problems during the execution of an activity and hence it would consume more time. The most likely time estimate is made in between the optimistic and the pessimistic estimates of time. Thus the three estimates of time have the relationship

$$t_o \le t_m \le t_p .$$

Practically speaking, neither the pessimistic nor the optimistic estimate may hold in reality and it is the most likely time estimate that is expected to prevail in almost all cases. Therefore, it is preferable to give more weight to the most likely time estimate.

We give a weight of 4 to most likely time estimate and a weight of 1 each to the pessimistic and optimistic time estimates. We arrive at a time estimate (t_e) as the weighted average of these estimates as follows:

$$t_e = \frac{t_o + 4\, t_m + t_p}{6}$$

Since we have taken 6 units (1 for t_p, 4 for t_m and 1 for t_o), we divide the sum by 6. With this time estimate, we can determine the project completion time as applicable for CPM.

Since PERT involves the average of three estimates of time for each activity, this method is very practical and the results from PERT will be have a reasonable amount of reliability.

1.6 Measure of Certainty

The 3 estimates of time are such that

$$t_o \leq t_m \leq t_p .$$

Therefore the range for the time estimate is $t_p - t_o$.

The time taken by an activity in a project network follows a distribution with a standard deviation of one sixth of the range, approximately.

i.e., The standard deviation = σ $= \dfrac{t_p - t_o}{6}$

And the variance will be $\sigma^2 = \left((t_p - t_o) /6 \right)^2$

The certainty of the time estimate of an activity can be analyzed with the help of the variance. The greater the variance, the more uncertainty in the time estimate of an activity.

Problem 1:

Two experts A and B examined an activity and arrived at the following time estimates.

Experts	Time estimates		
	t_o	T_m	T_p
A	4	6	8
B	4	7	10

Determine which expert is more certain about his estimates of time:

Solution:

Variance (σ^2) in time estimates $= \left((t_p - t_o) /6 \right)^2$

In the case of expert A, the variance = $\left((8 - 4) /6 \right)^2$ = 0.444

As regards expert B, the variance = $\left((10 - 4) /6 \right)^2$ = 1

So, the variance is less in the case of A. Hence, it is concluded that the expert A is more certain about his estimates of time.

1.7 Determination of Project Completion Time in PERT

Problem 2:

Find out the time required to complete the following project and the critical activities:

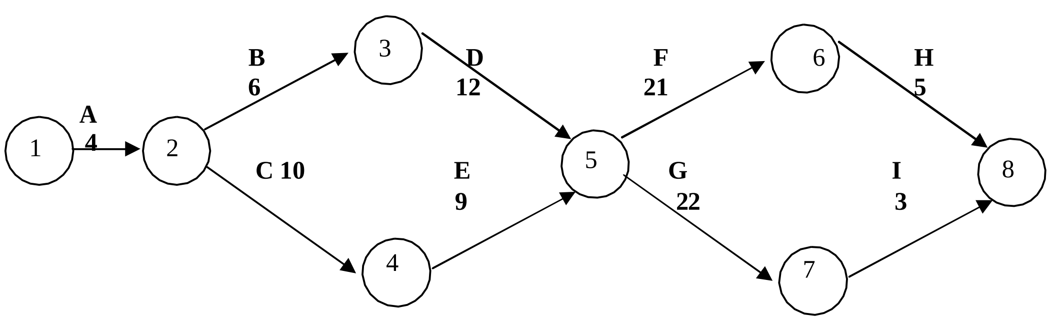
Activity	Predecessor Activity	Optimistic time estimate (t_o days)	Most likely time estimate (t_m days)	Pessimistic time estimate (t_p days)
A	-	2	4	6
B	A	3	6	9
C	A	8	10	12
D	B	9	12	15
E	C	8	9	10
F	D, E	16	21	26
G	D, E	19	22	25
H	F	2	5	8
I	G	1	3	5

Solution:

From the three time estimates t_p , t_m and t_o , calculate t_e for each activity. We obtain the following table:

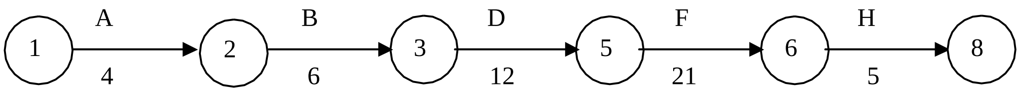
Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	$t_o + 4t_m + t_p$	Time estimate $t = \frac{t_o + 4t_m + t_p}{6}$
A	2	16	6	24	4
B	3	24	9	36	6
C	8	40	12	60	10
D	9	48	15	72	12
E	8	36	10	54	9
F	16	84	26	126	21
G	19	88	25	132	22
H	2	20	8	30	5
I	1	12	5	18	3

Using the single time estimates of the activities, we get the following network diagram for the project.



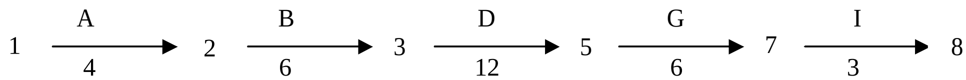
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

Path I



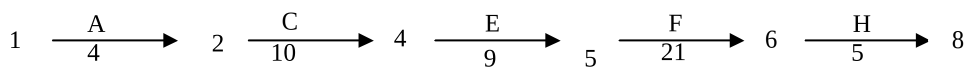
Time for the path: 4+6+12+21+5 = 48 days.

Path II



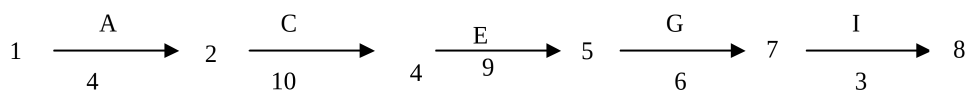
Time for the path: 4+6+12+ 6+3 = 31 days.

Path III



Time for the path: 4+10+9+ 21+5 = 49 days.

Path IV



Time for the path: 4+10+9+ 6+3 = 32 days.

Compare the times for the four paths.

Maximum of {48, 31, 49, 32} = 49.

We see that Path III has the maximum time.

Therefore the critical path is Path III. i.e., $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 8$.

The critical activities are A, C, E, F and H.

The non-critical activities are B, D, G and I. Project time (Also called project length) = 49 days.

Problem 3:

Find out the time, variance and standard deviation of the project with the following time estimates in weeks:

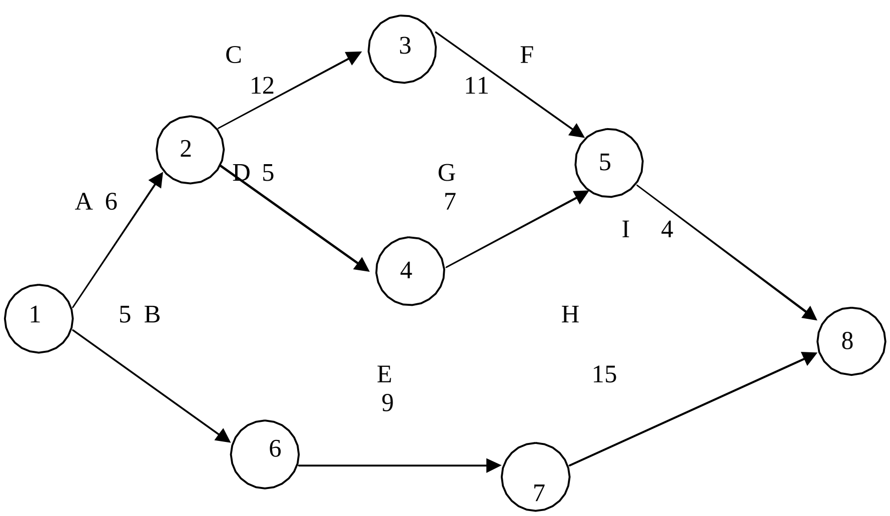
Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)
1-2	3	6	9
1-6	2	5	8
2-3	6	12	18
2-4	4	5	6
3-5	8	11	14
4-5	3	7	11
6-7	3	9	15
5-8	2	4	6
7-8	8	16	18

Solution:

From the three time estimates t_p , t_m and t_o , calculate t_e for each activity. We obtain the following table:

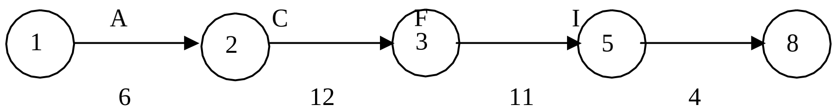
Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4\,t_m + t_p}{6}$
1-2	3	24	9	36	6
1-6	2	20	8	30	5
2-3	6	48	18	72	12
2-4	4	20	6	30	5
3-5	8	44	14	66	11
4-5	3	28	11	42	7
6-7	3	36	15	54	9
5-8	2	16	6	24	4
7-8	8	64	18	90	15

With the single time estimates of the activities, we get the following network diagram for the project.



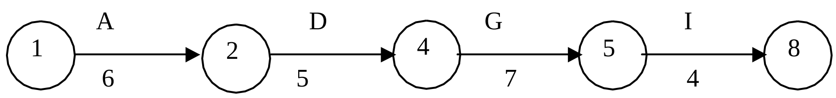
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



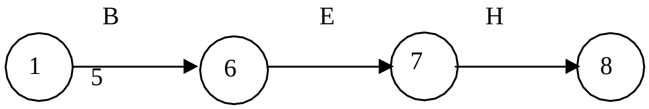
Time for the path: 6+12+11+4 = 33 weeks.

Path II



Time for the path: 6+5+7+ 4= 22 weeks.

Path III



Time for the path: 5+9+15 = 29 weeks.

Compare the times for the three paths. Maximum of {33, 22, 29} = 33.

It is noticed that Path I has the maximum time.

Therefore the critical path is Path I. i.e., $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 8$

The critical activities are A, C, F and I.

The non-critical activities are B, D,

G and H. Project time = 33 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)	Range ($t_p - t_o$)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance σ^2
A: $1 \rightarrow 2$	3	6	9	6	1	1
C: $2 \rightarrow 3$	6	12	18	12	2	4
F: $3 \rightarrow 5$	8	11	14	6	1	1
I: $5 \rightarrow 8$	2	4	6	4	2/3	4/9

Variance of project time (Also called Variance of project length) =

Sum of the variances for the critical activities = 1+4+1+ 4/9 = 58/9 Weeks.

Standard deviation of project time = √Variance = √58/9 = 2.54 weeks.

Problem 4

A project consists of seven activities with the following time estimates. Find the probability that the project will be completed in 30 weeks or less.

Activity	Predecessor Activity	Optimistic time estimate (t_o days)	Most likely time estimate (t_m days)	Pessimistic time estimate (t_p days)
A	-	2	5	8
B	A	2	3	4
C	A	6	8	10
D	A	2	4	6
E	B	2	6	10
F	C	6	7	8
G	D, E, F	6	8	10

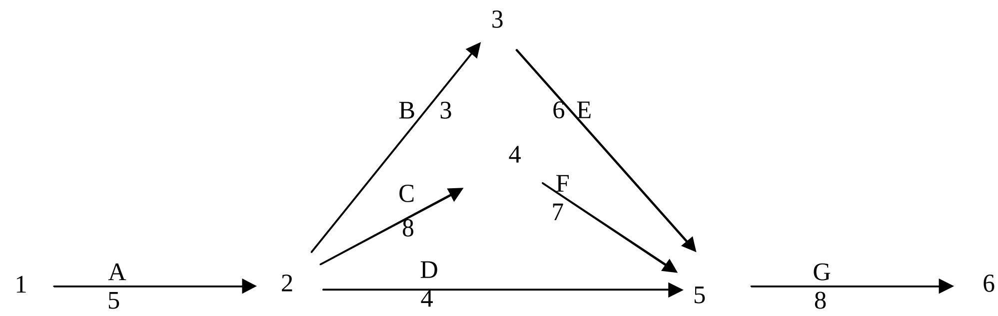
Solution:

From the three time estimates t_p , t_m and t_o
calculate t_e for each activity.

The results are furnished in the following table:

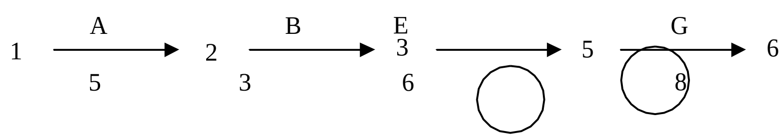
Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	t_o+ 4t_m+ t_p	Time estimate $t_e = \frac{t_o + 4\,t_m + t_p}{6}$
A	2	20	8	30	5
B	2	12	4	18	3
C	6	32	10	48	8
D	2	16	6	24	4
E	2	24	10	36	6
F	6	28	8	42	7
G	6	32	10	48	8

With the single time estimates of the activities, the following network diagram is constructed for the project.



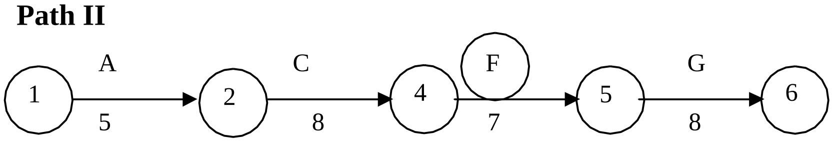
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



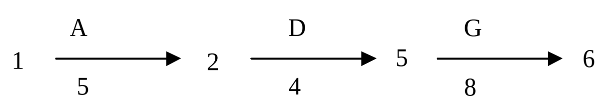
Time for the path: 5+3+6+8 = 22 weeks.

Path II



Time for the path: 5+8+7+ 8 = 28 weeks.

Path III



Time for the path: 5+4+8 = 17 weeks.

Compare the times for the three paths.

Maximum of {22, 28, 17} = 28.

It is noticed that Path II has the maximum time.

Therefore the critical path is Path II. i.e., $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$

The critical activities are A, C, F and G.

The non-critical activities are B, D and E. Project time = 28 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)	Range (t_p - t_o)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance σ^2
A: 1→2	2	5	8	6	1	1
C: 2→4	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$
F: 4→5	6	7	8	2	$\frac{1}{3}$	$\frac{1}{9}$
G: 5→6	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$

Standard deviation of the critical path = √2 = 1.414

The standard normal variate is given by the formula

$$Z = \frac{\textit{Given value of } t - \textit{Expected value of } t \text{ in } \textit{the critical path}}{\textit{SD for the critical path}}$$

So we get $Z = \frac{30 - 28}{1.414} = 1.414$

We refer to the Normal Probability Distribution Table. Corresponding to Z = 1.414, we obtain the value of 0.4207

We get 0.5 + 0.4207 = 0. 9207

Therefore the required probability is 0.92

i.e., There is 92% chance that the project will be completed before 30 weeks.

In other words, the chance that it will be delayed beyond 30 weeks is 8%

NORMAL DISTRIBUTION TABLE

Area Under Standard Normal Distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

1.8 Check your progress:

There are some activities to check you progress. Answer the followings:

1. The particular task performance in CPM is known

a) Dummy

b) Event

c) Activity

d) Contract

2. The critical path

a) Is a path that operates from the starting node to the end node

b) Is a mixture of all paths

c) Is the longest path

d) Is the shortest path

3. While scheduling a project by C.P.M.

a) A project is divided into various activities

b) Required time for each activity is established

c) A sequence of various activities is made according to their importance

d) All the above.

4. Which of the following statements is true?

a) The standard deviation of a project completion time is the sum of the standard deviations for the critical path activities

b) The variance of the time taken to complete an activity is equal to $(b - a)/6$

c) Three time estimates are necessary so that we can estimate the parameters of the Beta distribution

d) The critical path is the path with the largest probability of being completed on time.

5. An expected project completion time follows a normal distribution with a mean of 21 days and a standard deviation of 4 days. What is the probability that the project will be completed in a time between 22 to 25 days inclusive?

a) 0.0819

b) 0.7734

c) 0.8413

d) 0.2426

1.9 Summary

The critical path method (CPM) is an algorithm for scheduling a set of project activities. It is commonly used in conjunction with the program evaluation and review technique (PERT). A critical path is determined by identifying the longest stretch of dependent activities and measuring the time required to complete them from start to finish.

Program Evaluation and Review Technique (PERT) is a method used to examine the tasked that are in a schedule and determine a variation of the Critical Path Method (CPM). It analyzes the time required to complete each task and its associated dependencies to determine the minimum time to complete a project

The main difference between PERT and Critical Path is knowing how long a given task will take. With PERT, task durations are variable, hence the need to predict time using a model. Critical Path is more useful for projects where task length is easy to predict, such as a construction project or a large conference

Critical Path Analysis is commonly used with all forms of projects, including construction, aerospace and defense, software development, research projects, product development, engineering, and plant maintenance, among others. Any project with interdependent activities can apply this method of mathematical analysis. The first time CPM was used for major skyscraper development was in 1966 while constructing the former World Trade Center Twin Towers in New York City.

The essential technique for using CPM is to construct a model of the project that includes the following:

1. A list of all activities required to complete the project (typically categorized within a work breakdown structure),
2. The time (duration) that each activity will take to complete,
3. The dependencies between the activities and,
4. Logical end points such as milestones or deliverable items.

Using these values, CPM calculates the longest path of planned activities to logical end points or to the end of the project

1.10 Keywords:

***Critical activities*:** are those activities in any project that are to be completed within the given time otherwise project may be delayed.

Critical Path: is the path that comprises of all critical activities of a project.

Event: It is the beginning or the end of an activity. Events are represented by circles in a project network diagram

Predecessor Event: The event just before another event is called the predecessor event

Successor Event: The event just following another event is called the successor event

Network: A network is a series of related activities and events which result in an end product or service. The activities shall follow a prescribed sequence

1.11 Self Assessment Test:

1. Explain the terms: event, predecessor event, successor event, activity, dummy activity, and network.
2. Construct the network diagram for the following project:

Activity	Immediate Predecessor Activity
A	-
B	-
C	A
D	B
E	A
F	C, D
G	E
H	E
I	F, G
J	H, I

3. Explain the terms: critical path, critical activities.
4. The following are the time estimates and the precedence relationships of the activities in a project network:

Activity	IMMEDIATE Predecessor Activity	time estimate (weeks)
A	-	4
B	-	7
C	-	3
D	A	6
E	B	4
F	B	7
G	C	6
H	E	10
I	D	3
J	F, G	4
K	H, I	2

Draw the project network diagram. Determine the critical path and the project completion time.

5. Draw the network diagram for the following project. Determine the time, variance and standard deviation of the project.

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	12	14	22
B	-	16	17	24
C	A	14	15	16
D	A	13	18	23
E	B	16	18	20
F	D,E	13	14	21
G	C,F	6	8	10

6. Consider the following project with the estimates of time in weeks:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	-	3	5	7
C	A	5	6	13
D	A	4	8	12
E	B,C	5	6	13
F	D,E	6	8	14

Find the probability that the project will be completed in 27 weeks.

1.12 Answers to check your progress:

1. (c) activity
2. © longest path
3. (d) all the above
4. © Three time estimates are necessary so that we can estimate the parameters of the Beta distribution
5. (d) 0.2426

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